Logic of Subtyping

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Abstract

We introduce new modal logical calculi that describe subtyping properties of Cartesian product and disjoint union type constructors as well as mutually-recursive types defined using those type constructors.

Basic Logic of Subtyping **S** extends classical propositional logic by two new binary modalities $\otimes$ and $\oplus$. An interpretation of **S** is a function that maps standard connectives into set-theoretical operations (intersection, union, and complement) and modalities into Cartesian product and disjoint union type constructors. This allows **S** to capture many subtyping properties of the above type constructors. We also consider logics $\mathbf{S}_\rho$ and $\mathbf{S}^\omega_\rho$ that incorporate into **S** mutually-recursive types over arbitrary and well-founded universes correspondingly.

The main results are completeness of the above three logics with respect to appropriate type universes. In addition, we prove Cut elimination theorem for **S** and establish decidability of **S** and $\mathbf{S}^\omega_\rho$.

Key words: subtype, Curry-Howard isomorphism, proposition-as-type, non-standard logic, recursive types, cut-elimination
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1 Introduction

1.1 Logical connectives as set operations

We are interested in the use of logical connectives to describe properties of set and type operations. Historically, there have been two major ways to inter-

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pret logical connectives as such operations: set semantics and Curry-Howard isomorphism.

According to the set semantics, every propositional formula is evaluated to a subset of a given universe U and propositional connectives conjunction $\wedge$, disjunction $\vee$, and negation $\neg$ are identified with set operations intersection $\cap$, union $\cup$, and complement $\complement_U$ correspondingly. It is easy to see that a formula is provable in the classical propositional logic if and only if it is evaluated to the entire universe U under any set semantics.

Under Curry-Howard isomorphism (Curry [1934, 1942], Curry and Feys [1958], Howard [1980]), propositional formulas are evaluated to types and connectives $\wedge, \vee$, and $\rightarrow$ are interpreted as Cartesian product $\times$, disjoint union $+$, and constructive function $\mapsto$ type constructors. It could be shown that a formula is provable in intuitionistic propositional logic (Int) if and only if it is evaluated to an inhabited type. Intuitionistic logic describes properties of Cartesian product, disjoint union, and function type constructors that can be expressed in terms of type inhabitness.

Since the list of possible type constructors is not limited to product, disjoint union, and function, one can raise a question about logical principles describing behavior of other type constructors. For example, list, partial object [Smith, 1995] and squash [Constable et al., 1986] types can be viewed as modalities while inductive and co-inductive constructors (Mendler [1991] and Coquand and Paulin [1990]) may be considered as quasi-quantifiers. Kopylov and Nogin [2001] established that modal logic of squash operator is, in fact, Lax Logic [Fairtlough and Mendler, 1997]. If instead of types one considers languages then logical connectives corresponding to product and star operations are described by Interval Temporal Logic (Moszkowski and Manna [1984]).

1.2 Logic of Subtyping

In this paper we propose a logical system **S** that describes subtyping properties of Cartesian product and disjoint union. This system is an extension of the classical logic by two binary modalities $\oplus$ and $\otimes$, corresponding to Cartesian product and disjoint union type constructors. Logic **S** is not the first logical calculus aimed at axiomatization of subtyping. Meyer [2000] emphasizes that, developed with philosophical reasons in mind, Logic of Entailment Meyer and Routley [1972], Routley and Meyer [1972], can be viewed as a logic of subtyping. Mitchell [1988] axiomatized subtyping relation for polymorphic types and proved completeness of this axiomatization with respect to Girard's Fairtlough and Mendler [1997] system **F** as well as its completeness with respect to a class of models. This axiomatization only deals with $\rightarrow$ and $\forall$, the type constructors present in the system **F**. Mitchell's subtyping relation is proven in Tiuryn and Urzyczyn [1996] to be undecidable. Longo, Milsted, and Soloviev Longo et al. [1995] proposed to treat subtyping predicate as a sequent $\vdash$. The authors designed a sequent calculus, which they called "logic of subtyping", and proved its equivalence to Mitchell's systems. Later, in Longo et al. [2000], they have started to call this approach "subtyping as entailment". Natural deduction and Gentzen-style calculi for this logic of subtyping is described in Tiuryn [2001]. Valentini and Viale [2002] investigated a similar logic of subtyping that includes intersection types. They treat function type as a binary modality and prove completeness of their logic with respect to applicative structures.

The main advantage of the system **S** is that it, unlike earlier calculi, treats subtyping as an operation, not a relation. Thus, it allows nested occurrences of subtyping in formulas, making logic more expressive and showing closer connection between subtyping and implication. For example, the following **S**-tautology: $\phi_1 \otimes \phi_2 \to (\psi_1 \otimes \psi_2 \to \phi_1 \otimes \psi_2)$, under the defined later semantics, states that any element of a two different Cartesian products also belongs to a "cross-over" of these products. Among less trivial properties that can be expressed in **S** is the following property of Cartesian product covers:

Proposition 1 *Let I be a finite set T, S be two arbitrary sets, and $\{T_i\}_{i\in I}$, $\{S_i\}_{i\in I}$ be two families of sets then $T \times S \subseteq \bigcup_{i\in I}(T_i \times S_i)$ if and only if $\forall J \subseteq I \left(T \subseteq \bigcup_{i\in J} T_i \;\; or \;\; S \subseteq \bigcup_{i\in I\setminus J} S_i\right)$.*

It is formalized in **S** by the inference rule (XP), which will be defined later. Remarkably, (XP) is the only rule of **S** related to Cartesian product.

1.3 Recursive Types

Recursive type is an arbitrary solution T of a fixed point type equation $T = f(T)$. A family of mutually recursive types is an arbitrary solution of a system of type equations $\{T_i = f_i(T_1, \ldots, T_n) | i = 1, \ldots, n\}$. Existence of such types for monotone functions f_i could be established using Tarski [1955] fixed point theorem for complete lattices or Banach fixed-point theorem for contractive maps on complete metric spaces (see MacQueen et al. [1986]). The case of non-monotone functions is considered in Cardone and Coppo [1991]. Recursive types have been used to provide semantics for programming languages and a foundation for automated theorem proving Morris [1968], Huet [1976], Constable and Mendler [1985], Coquand and Paulin [1990], Mendler [1991]. Most commonly, the least and the greatest solutions of the above type equations are considered. They are known as inductive and co-inductive types. This paper studies subtyping logic of arbitrary recursive types. Logics of inductive and co-inductive types would constitute extensions of the logic of arbitrary recursive types.

Subtyping relation for recursive types was studied in Amadio and Cardelli Amadio and Cardelli [1993], where a set of type rules and a subtype checking algorithm for a simply typed λ-calculus is given. These results have been later extended in Kozen et al. [1995], Brandt and Henglein [1998], and Jim and Palsberg [1999]. Ghelli [1993] has shown that subtyping algorithm for a second order system can not be expanded to cover recursive types. Later, in Colazzo and Ghelli [1999], they gave such algorithm for a system with bounded second order polymorphism. The are several points in which this paper is different from the previous works on subtyping of recursive types. First, we consider recursive types built using type constructors corresponding to logical connectives $\otimes$, $\oplus$, and $\rightarrow$ of the defined above Logic of Subtyping **S**, not a function type constructor. Second, we deal with families of mutually-recursively defined types. Finally, we do not introduce into the language of our logic **S** any new constructors that would represent recursive types. Instead, we assume that propositional variables of our logic satisfy some system ρ of fixed point equations. Thus, instead of defining one Subtyping Logic of Mutually Recursive Types, we will specify a family $\mathbf{S}_\rho$ of such logics, where ρ ranges over all possible systems of fixed point equations. Of course, the last difference is just a matter of notation, but it results in a more elegant presentation of the logic.

1.4 Type Universe

Introduced in this paper logics of subtyping do not describe subtyping properties of any particular type system. Instead, they describe common properties of rather general class of type universes. In other words, we are interested in the subtyping tautologies that are true in all models of type systems. A universe is a set (of terms) with defined on it injections $inl(x)$, $inr(x)$, and $pair(x, y)$. The only condition implosed on those functions is that $inl(x) \neq inr(y)$ for any two terms x and y. Any model of set theory with standard definitions of *inl*, *inr*, and *pair* satisfies this definition. So do models of different type theories defined as a set of syntactical terms (expressions) with an equivalence relation on it. At the same time, some type theories, such as Nuprl (Constable et al. [1986]), do not require their models to have a uniform equality relation. Instead, each type has its own equality. Although, of course, one can define a uniform equality as the intersection of all equalities, with respect to such uniform equality functions $inl(x)$, $inr(x)$, and $pair(x, y)$ may not be injections. If this is the case then such model is outside of the class of type universes considered in this paper.

Finally, a type we will mean any subset of the universe. This allows the set of types to be closed under operations of complement and union which is important given our interpretation of the connective $\to$. Some type systems consider a more narrow class of types. From the viewpoint of such a system, our results are still meaningful, because our extra types could be viewed as just a tool for expressing properties of "lagidimite" types.

1.5 Well-Foundness

In addition to studying the minimal subtyping logic of mutually-recursive types that describes properties of all possible type universes, we also consider logic of well-founded universes. We call a universe well-founded if any element in this universe has a finite "structure". Formal definition will be given later. Standard encoding of type operations into ZF produces well-founded type universe. Another example of a well-founded type universe is the universe of expressions built from atoms using operations *pair*, *inl*, and *inr* and syntactical equality of terms. On the other hand, any universe that includes so-called "streams" is not well-founded.

We will show that logic **S** is complete not only with respect to the class of all universes, but also with respect to the class of well-founded universes. At the same time, logic $\mathbf{S}_\rho$ has to be extended by an extra well-foundness rule to make it complete with respect to well-founded universes.

2 Logics of Subtyping

2.1 Syntax

By a modal propositional formula we mean any expression built from propositional variables $p, q, r, \dots$, using binary connectives $\to$, $\otimes$, $\oplus$ and 0-arity connective $\bot$. As usual, $\top$, $\neg\phi$, $\phi \vee \psi$, and $\phi \wedge \psi$ are considered to be abbreviations for $\bot \to \bot$, $\phi \to \bot$, $\neg\phi \to \psi$, and $\neg(\neg\phi \vee \neg\psi)$ correspondingly. In addition, meta negation $\bar{\phi}$ of a modal propositional formula ϕ is a syntactical operation that is defined as following: if $\phi \equiv \neg\psi$ for some propositional formula ψ then $\bar{\phi}$ is ψ, otherwise, $\bar{\phi}$ is $\neg\phi$. Finally, a sequent is an arbitrary pair of finite multisets of modal propositional formulas. Sequent formed by multisets Γ and Δ is denoted by $\Gamma \Rightarrow \Delta$.

The syntax of subtyping logics, as defined above, does not include any logical connectives corresponding to recursive type constructors. Instead, recursive types are incorporated into the logic as a set of modal fixed point equations $\{p_i \equiv \phi_i\}_{i\in I}$ for some finite set I. Such set of equations will be called a *recursion*. Formal definition of recursion goes as follows:

Definition 1 *Recursion is an arbitrary function from propositional variables into modal propositional formulas. Recursion ρ is finite if $\rho(p) = p$ for all but finitely many different propositional variables p.*

Of course, not every system of fixed point equations is consistent, or, in other words, has a type solution. For example, this one: $\{p \equiv \neg p\}$ has not. As it will follow from our completeness theorem, in order for such a solution to exist, it is sufficient for ρ to be positive in the following sense:

Definition 2 *An occurrence of a propositional variable in a propositional formula is positive if either this occurrence is within the scope of $\oplus$ or $\otimes$ connective or it is not within the scope of an $\oplus$ or an $\otimes$ connective, but is in the premises of an even number of implications.*

Recursion ρ will be called *positive* if for any propositional variable p, formula $\rho(p)$ has only positive occurrences of propositional variables. Note that identity function *id* on propositional formulas is a finite positive recursion.

2.2 Semantics

Definition 3 *A universe U is a quadruple $\langle Term, pair, inl, inr\rangle$, where $Term$ is an arbitrary non-empty set, pair is an injection that maps any two elements of set $Term$ into an element of the same set, and inl and inr are two injections from $Term$ into $Term$. It is assumed that the last two functions also satisfy the following condition: $\forall x, y \in Term\ (inl(x) \neq inr(y))$.*

Elements of the set $Term$ are called terms of the universe U. An arbitrary subset of $Term$ will be called a type of the universe U. In addition to standard set-theoretical operations on types, such as union, intersection, and complement, we define operations Cartesian product $S \times T = \{pair(s,t) \mid s \in S, t \in T\}$ and disjoint union $S + T = \{inl(s) \mid s \in S\} \cup \{inr(t) \mid t \in T\}$.

Definition 4 *For any two terms t and s of a universe U, we say that s is a greatest subterm of t, written as $t > s$, if $t = inl(s)$, $t = inr(s)$, $t = pair(r,s)$, or $t = pair(s,r)$), for some term r. Let relation subterm $\succ$ be the transitive closure of the relation $>$.*

Definition 5 *A universe is called well-founded if any chain of its terms $t_1 \succ t_2 \succ t_3 \succ \ldots$ is finite. By $rank(t_1)$ we mean the maximal length among all chains starting at t_1.*

Definition 6 *Let U be a type universe. A valuation over U is an arbitrary function from propositional variables into types of universe U.*

Definition 7 *For any propositional formula ϕ and any valuation $*$ over a type universe U, we define type ϕ^* of the universe U recursively as*

(1) $\bot^* = \varnothing$,
(2) $p^* = *(p)$,
(3) $(\psi \to \chi)^* = \complement(\psi^*) \cup \chi^*$,
(4) $(\psi \oplus \chi)^* = \psi^* + \chi^*$,
(5) $(\psi \otimes \chi)^* = \psi^* \times \chi^*$.

Let ρ be a recursion. We say that valuation $*$ is ρ-sound if for any propositional variable p, $(\rho(p))^* = p^*$.

Definition 8 *For any sequent $\Gamma \Rightarrow \Delta$ and any recursion ρ, let $\models_\rho \Gamma \Rightarrow \Delta$ iff for any ρ-sound valuation $*$ over an arbitrary universe, $\bigcap_{\gamma\in\Gamma} \gamma^* \subseteq \bigcup_{\delta\in\Delta} \delta^*$, where an intersection of an empty family of types is the entire universe U. Let $\models^\omega_\rho \Gamma \Rightarrow \Delta$ mean that the same is true for an arbitrary well-founded universe.*

2.3 Inference Rules

Here we present subtyping logics as a Gentzen-style (sequential) calculus. Hilbert-style version of **S** is described later.

Definition 9 *For any recursion ρ, Logic of Subtyping $\mathbf{S}_\rho$ is the extension of the classical sequential logic*[1] *by the following four inference rules:*

$$\frac{\{\phi_i\}_{i\in I} \Rightarrow \{\psi_j\}_{j\in J} \quad \{\chi_i\}_{i\in I} \Rightarrow \{\eta_j\}_{j\in J} \quad |I|>0}{\{\phi_i \oplus \chi_i\}_{i\in I} \Rightarrow \{\psi_j \oplus \eta_j\}_{j\in J}} \text{ (XU)},$$

$$\frac{\forall J' \subseteq J[\{\phi_i\}_{i\in I} \Rightarrow \{\psi_j\}_{j\in J'} \text{ or } \{\chi_i\}_{i\in I} \Rightarrow \{\eta_j\}_{j\in J\setminus J'}] \qquad |I|>0}{\{\phi_i \otimes \chi_i\}_{i\in I} \Rightarrow \{\psi_j \otimes \eta_j\}_{i\in J}} \text{ (XP)},$$

$$\frac{\Gamma, p \Rightarrow \Delta}{\Gamma, \rho(p) \Rightarrow \Delta} \text{ (LR)}, \qquad \frac{\Gamma \Rightarrow \Delta, p}{\Gamma \Rightarrow \Delta, \rho(p)} \text{ (RR)}.$$

If the (XU) rule is quite simple, the same hardly could be said about the (XP) rule. Each instance of this rule will have as many hypotheses as there are subsets of the set J. In fact, this rule probably should be called rule schema because for any conclusion of this rule there are $2^{2^{|J|}}$ possible sets of hypotheses from which this conclusion could have been deducted. This is

[1] See, for example, propositional fragment of system **G1c** in Troelstra and Schwichtenberg [2000].

because for every subset J' of J only one of statements $\{\phi_i\}_{i\in I} \Rightarrow \{\psi_j\}_{j\in J'}$ and $\{\chi_i\}_{i\in I} \Rightarrow \{\eta_j\}_{j\in J\setminus J'}$ needs to be on the hypotheses list. The rule is a formalization in our language of Proposition 1.

Definition 10 *For any recursion ρ, Subtyping Logic of Well-Founded Universes $\mathbf{S}^\omega_\rho$ is the extension of $\mathbf{S}_\rho$ by the following inference rule:*

$$\frac{\phi \Rightarrow \phi \otimes \psi, \psi \otimes \phi, \phi \oplus \phi}{\phi \Rightarrow} \text{ (WF)}.$$

Basic logic of subtyping $\mathbf{S}$ is the subtyping logic that does not include (LR) and (RR) rules. Alternatively, it can be viewed as $\mathbf{S}_{id}$, where id is the identity recursion. Similarly, $\mathbf{S}^\omega$ is defined as $\mathbf{S}^\omega_{id}$. Later in this paper (Corollary 2) we will establish that $\mathbf{S}^\omega = \mathbf{S}$.

Theorem 1 *For any sequent $\Gamma \Rightarrow \Delta$ and any finite positive recursion ρ,*

(1) $\mathbf{S}_\rho \vdash \Gamma \Rightarrow \Delta \iff\ \models_\rho \Gamma \Rightarrow \Delta$,
(2) $\mathbf{S} \vdash \Gamma \Rightarrow \Delta \iff\ \models^\omega_{id} \Gamma \Rightarrow \Delta$,
(3) $\mathbf{S}^\omega_\rho \vdash \Gamma \Rightarrow \Delta \iff\ \models^\omega_\rho \Gamma \Rightarrow \Delta$.

Soundness. Induction on the size of the derivation. Soundness of rules of Classical Logic is trivial. Let us consider non-classical rules.

(XP). Assume that the valuations of the rule premises are true. We will show that for any term t, if $t \in \bigcap_{i\in I}(\phi_i \otimes \chi_i)^*$ then there is $j_0 \in J$ such that $t \in (\psi_{j_0} \otimes \eta_{j_0})^*$. If $t \in \bigcap_{i\in I}(\phi_i \otimes \chi_i)^*$ then $t \in \bigcap_{i\in I}(\phi_i^* \times \chi_i^*)$. Since I is not empty, $t \equiv pair(t_1, t_2)$ for some terms t_1 and t_2 such that $t_1 \in \phi_i^*$ and $t_2 \in \chi_i^*$ for every $i \in I$. Thus, $t_1 \in \bigcap_{i\in I} \phi_i^*$ and $t_2 \in \bigcap_{i\in I} \chi_i^*$. Let $J' = \{j \in J \mid t_2 \in \eta_j^*\}$. The valuation of the (XP) rule assumptions implies that either $\bigcap_{i\in I} \phi_i^* \subseteq \bigcup_{j\in J'} \psi_j^*$ or $\bigcap_{i\in I} \chi_i^* \subseteq \bigcup_{j\in J\setminus J'} \eta_j^*$. Let us consider those two cases separately:

(1) Statement $t_1 \in \bigcap_{\in I} \phi_i^*$ implies $t_1 \in \bigcup_{j\in J'} \psi_j^*$. Thus, there is $j_0 \in J'$ such that $t_1 \in \psi_{j_0}^*$. At the same time, by the definition of J', $t_2 \in \eta_{j_0}^*$. Hence, $t = pair(t_1, t_2) \in \psi_{j_0}^* \times \eta_{j_0}^* = \psi_{j_0} \otimes \eta_{j_0}^*$.
(2) Since $t_2 \in \bigcap_{i\in I} \chi_i^*$, we have $t_2 \in \bigcup_{j\in J\setminus J'} \eta_j^*$. Thus, there is $j_0 \in J\setminus J'$ such that $t_2 \in \eta_{j_0}^*$. By the definition of J', such j_0 would have to belong to J'. Contradiction.

(XU). Assume that the valuations of the rule premises are true. We will show that for any term t, if $t \in \bigcap_{i\in I}(\phi_i \oplus \chi_i)^*$ then there is $j_0 \in J$ such that $t \in (\psi_{j_0} \oplus \eta_{j_0})^*$. First of all, $t \in \bigcap_{i\in I}(\phi_i \oplus \chi_i)^*$ implies that $t \in (\phi_i \oplus \chi_i)^*$ for every $i \in I$. Since I is not empty, either $t \equiv inl(u)$ or $t \equiv inr(u)$ for some term u such that $\forall i(u \in (\phi_i)^*)$ or $\forall i(u \in (\chi_i)^*)$ correspondingly. Let us consider the first case. Hence, $u \in \bigcap_{i\in I} \phi_i^*$. Valuation of the first assumption of (XU) rule is $\bigcap_{i\in I}\phi_i^* \subseteq \bigcup_{j\in J}\psi_j^*$. Thus, $u \in \bigcup_{j\in J}\psi_j^*$. Therefore, $u \in \psi_{j_0}^*$ for some $j_0 \in J$. Finally, $t \equiv \mathit{inl}(u) \in \psi_{j_0}^* + \eta_{j_0}^* = (\psi_{j_0} \oplus \eta_{j_0})^*$. The second case is similar.

(LR). Since $*$ is a ρ-sound valuation, $(\rho(p))^* = (p)^*$. Therefore, $\bigcap_{\gamma\in\Gamma}\gamma^* \cap (\rho(p))^* \subseteq \bigcup_{\delta\in\Delta}\delta^* \cup p^*$. Rule (RR) can be handled similarly.

(WF). Assume that for some valuation $*$ and some modal propositional formulas ϕ, ψ, type ϕ^* is not empty and $\phi^* \subseteq (\phi^* \times \psi^*) \cup (\psi^* \times \phi^*) \cup (\phi^* + \phi^*)$. Let t_0 be an element of ϕ^* of the lowest rank. The subtyping statement above implies that $t = \mathit{pair}(t_1, t_2)$ for some $t_1 \in \phi^*$, $t = \mathit{pair}(t_1, t_2)$ for some $t_2 \in \phi^*$, $t = \mathit{inl}(t_1)$ for some $t_1 \in \phi^*$, or $t = \mathit{inr}(t_2)$ for some $t_2 \in \phi^*$. Any of those statements contradicts to the minimality of rank of t among elements of ϕ^*. $\square$

Completeness will be established later.

Lemma 1 $\mathbf{S}_\rho \vdash \psi_1 \otimes \psi_2 \Rightarrow \phi_1 \otimes \phi_2, \bar{\phi_1} \otimes \psi_2, \psi_1 \otimes \bar{\phi_2}$*, for any propositional formulas* $\phi_1, \phi_2, \psi_1, \psi_2$.

Proof. By (XP) rule, the above sequent is provable if the following eight statements are true: (1) $\mathbf{S}_\rho \vdash \psi_1 \Rightarrow \phi_1, \bar{\phi_1}, \psi_1$ or $\mathbf{S}_\rho \vdash \psi_2 \Rightarrow$, (2) $\mathbf{S}_\rho \vdash \psi_1 \Rightarrow \phi_1, \bar{\phi_1}$ or $\mathbf{S}_\rho \vdash \psi_2 \Rightarrow \bar{\phi_2}$, (3) $\mathbf{S}_\rho \vdash \psi_1 \Rightarrow \phi_1, \psi_1$ or $\mathbf{S}_\rho \vdash \psi_2 \Rightarrow \psi_2$, (4) $\mathbf{S}_\rho \vdash \psi_1 \Rightarrow \phi_1$ or $\mathbf{S}_\rho \vdash \psi_2 \Rightarrow \psi_2, \bar{\phi_2}$, (5) $\mathbf{S}_\rho \vdash \psi_1 \Rightarrow \bar{\phi_1}, \psi_1$ or $\mathbf{S}_\rho \vdash \psi_2 \Rightarrow \phi_2$, (6) $\mathbf{S}_\rho \vdash \psi_1 \Rightarrow \bar{\phi_1}$ or $\mathbf{S}_\rho \vdash \psi_2 \Rightarrow \phi_2, \bar{\phi_2}$, (7) $\mathbf{S}_\rho \vdash \psi_1 \Rightarrow \psi_1$ or $\mathbf{S}_\rho \vdash \psi_2 \Rightarrow \phi_2, \psi_2$, (8) $\mathbf{S}_\rho \vdash \psi_1 \Rightarrow$ or $\mathbf{S}_\rho \vdash \psi_2 \Rightarrow \phi_2, \psi_2, \bar{\phi_2}$. We are left to notice that, since $\mathbf{S}_\rho$ is an extension of the classical propositional logic, in statements 1, 2, 3, 5, and 7 the left disjunct is true and in statements 4, 6, and 8 the right disjunct is true. $\square$

Lemma 2 $\mathbf{S}_\rho \vdash \psi_1 \oplus \psi_2, \neg(\phi_1 \oplus \phi_2) \Rightarrow \bar{\phi_1} \oplus \bar{\phi_2}$*, for any propositional formulas* $\phi_1, \phi_2, \psi_1, \psi_2$.

Proof. The following two sequents are provable in the classical fragment of $\mathbf{S}_\rho$: $\psi_1 \Rightarrow \phi_1, \bar{\phi_1}$ and $\psi_2 \Rightarrow \phi_2, \bar{\phi_2}$. Thus, by (XU) rule, $\mathbf{S}_\rho \vdash \psi_1 \oplus \psi_2 \Rightarrow \phi_1 \oplus \phi_2, \bar{\phi_1} \oplus \bar{\phi_2}$. Therefore, by (LI) rule, $\mathbf{S}_\rho \vdash \psi_1 \oplus \psi_2, \neg(\phi_1 \oplus \phi_2) \Rightarrow \bar{\phi_1} \oplus \bar{\phi_2}$. $\square$

3 Kripke Semantics

In this section we prove completeness theorem for subtyping logics with respect to a class of Kripke models. Later those Kripke models will be transformed into type universes to prove type completeness. Under this transformation, worlds of a Kripke model will be mapped into terms of a universe. To prove completeness with respect to Kripke models we first define a more general notion of a Kripke structure and show that any Kripke structure contains an "embedded" Kripke model. Second, we finish completeness proof by constructing a canonical Kripke structure.

3.1 Kripke Structure

Definition 11 *Kripke structure is an arbitrary four-tuple $\langle W, R, L, \|\cdot\|\rangle$, where W is a finite set of "worlds", R is a binary relation ("directed graph") on W, $L : R \mapsto \{\pi_1, \pi_2, \sigma_1, \sigma_2\}$ is a function that assigns labels to the edges of the graph, $\|\cdot\| : W \mapsto 2^{Var}$ is a function that for every vertex of the graph specifies a set of propositional variables. We write $wR_\alpha v$ if $(w, v) \in R$ and $L(w, v) = \alpha$.*

Definition 12 *Let W be the set of worlds of an arbitrary Kripke structure. We define subsets Δ_n, Π_n, Σ_n, and Δ of W as follows:*

(1) $\Delta_0 = \varnothing$,
(2) $\Pi_n = \{w \mid \exists u, v(wR_{\pi_1}u \wedge wR_{\pi_2}v) \to \exists u, v \in \Delta_n(wR_{\pi_1}u \wedge wR_{\pi_2}v)\}$
(3) $\Sigma_n = \{w \mid \exists u(wR_{\sigma_1}u \vee wR_{\sigma_2}u) \to \exists u \in \Delta_n(wR_{\sigma_1}u \vee wR_{\sigma_2}u)\}$
(4) $\Delta_{n+1} = \Delta_n \cup (\Pi_n \cap \Sigma_n)$,
(5) $\Delta = \bigcup_n \Delta_n$.

Elements of set Δ will be referred to as *well-founded worlds* of the Kripke structure. The set of all worlds that are not well-founded is denoted by Ω. Informally, a world is well-founded if it can be decomposed into atomic worlds. A Kripke structure is called well-founded if every world of this structure is well-founded.

Definition 13 *Function $h : W \mapsto W'$ is a homomorphism between Kripke structures $\langle W, R, L, \|\cdot\|\rangle$ and $\langle W', R', L', \|\cdot\|'\rangle$ if for any world w and any label α, $wR_\alpha u$ implies $h(w)R'_\alpha h(u)$ and, in addition, $\|h(w)\|' = \|w\|$.*

3.2 Kripke Model

Definition 14 *Kripke model is a Kripke structure $\langle W, R, L, \|\cdot\|\rangle$ that satisfies the following additional conditions:*

(1) $\langle W, R\rangle$ is a, not necessary finite, DAG.
(2) Any two edges leaving the same vertex have different labels.
(3) Any vertex either has both π_1- and π_2- labeled outgoing edges or none of them.
(4) Any vertex can not have both σ_1- and σ_2- outgoing edges.

Definition 15 *Let w be a node of a Kripke model. By $\alpha(w)$ we mean the* *unique node v such that $wR_\alpha v$, if such node exists.*

Definition 16 *For any node w of a Kripke model and an arbitrary propositional formula ϕ, relation $w \Vdash \phi$ is defined by induction on complexity of formula ϕ:*

(1) $w \nVdash \bot$,
(2) $w \Vdash p$ *if and only if* $p \in \|w\|$,
(3) $w \Vdash \phi_1 \to \phi_2$ *if and only if* $w \nVdash \phi_1$ *or* $w \Vdash \phi_2$,
(4) $w \Vdash \phi_1 \otimes \phi_2$ *if and only if* $\pi_1(w) \Vdash \phi_1$ *and* $\pi_2(w) \Vdash \phi_2$,
(5) $w \Vdash \phi_1 \oplus \phi_2$ *if and only if* $\sigma_1(w) \Vdash \phi_1$ *or* $\sigma_2(w) \Vdash \phi_2$.

Definition 17 *Kripke model is ρ-sound if for any world w of this model and any propositional variable p, $w \Vdash p$ if and only if $w \Vdash \rho(p)$.*

Definition 18 *For any sequent $\Gamma \Rightarrow \Delta$ and any recursion ρ, let $\Vdash_\rho \Gamma \Rightarrow \Delta$ mean that for any world w of any ρ-sound Kripke model if $w \Vdash \gamma$ for all $\gamma \in \Gamma$ then $w \Vdash \delta$ for at least one $\delta \in \Delta$. In addition, let $\Vdash^F_\rho \Gamma \Rightarrow \Delta$ mean that the same is true for any finite Kripke model*[2] *.*

Theorem 2 *For any sequent $\Gamma \Rightarrow \Delta$ and any finite recursion ρ,*

(1) $\mathbf{S}_\rho \vdash \Gamma \Rightarrow \Delta \quad \Longleftrightarrow \quad \Vdash_\rho \Gamma \Rightarrow \Delta$,
(2) $\mathbf{S} \vdash \Gamma \Rightarrow \Delta \quad \Longleftrightarrow \quad \Vdash^F_{id} \Gamma \Rightarrow \Delta$,
(3) $\mathbf{S}^\omega_\rho \vdash \Gamma \Rightarrow \Delta \quad \Longleftrightarrow \quad \Vdash^F_\rho \Gamma \Rightarrow \Delta$.

Proof. Soundness ($\Rightarrow$) could be shown similarly to soundness part of Theorem 1. Completeness ($\Leftarrow$) will be established later. □

Corollary 1 *For any finite recursion ρ, logics $\mathbf{S}$ and $\mathbf{S}^\omega_\rho$ are decidable.*

Corollary 2 *For any sequent $\Gamma \Rightarrow \Delta$, $\mathbf{S} \vdash \Gamma \Rightarrow \Delta$ if and only if $\mathbf{S}^\omega \vdash \Gamma \Rightarrow \Delta$.*

3.3 Embedded Models

Definition 19 *Embedded model of a Kripke structure K is a pair $\langle M, h\rangle$, where M is a Kripke model and h is a homomorphism of M into K.*

Definition 20 *Let $\mu = \langle M, h\rangle$ be an embedded model of a Kripke structure K, where $M = \langle W', R', L', \|\cdot\|'\rangle$ and $K = \langle W, R, L, \|\cdot\|\rangle$. A world $w \in W'$ is π-complete if*

$$\exists x, y \in W(h(w)R_{\pi_1}x \wedge h(w)R_{\pi_2}y) \to \exists u, v \in W'(wR'_{\pi_1}u \wedge wR'_{\pi_2}v)$$

[2] Well-founded type universes will correspond to *finite* Kripke models.

the same world is σ-complete if

$$\exists x \in W(h(w)R_{\sigma_1}x \vee h(w)R_{\sigma_2}x) \rightarrow \exists u \in W'(wR'_{\sigma_1}u \vee wR'_{\sigma_2}u)$$

The embedded model is complete if each world of this model is π- and σ-complete.

Clearly any embedded tree can be made complete by expanding it (possibly infinitely many times):

Lemma 3 *For any world w of a Kripke structure K there is a complete embedded model $\langle M, h\rangle$ of K and a world v of model M such that $h(v) = w$.* □

In some cases the embedded complete tree might be made finite. Below, Δ_n refers to the set defined in Definition 12:

Lemma 4 *For any well-founded world $w \in \Delta_n$ of a Kripke structure K there is a finite complete embedded model $\langle M, h\rangle$ of K and a world v of model M such that $h(v) = w$.*

Proof. Induction on n. Base case is true because Δ_0, by definition, is empty. Assume $w \in \Delta_{n+1}$. Let $k \leq n$ be the smallest k such that $w \in \Delta_{k+1}$. Thus, $w \in \Pi_k$ and $w \in \Sigma_k$. Note that $w \in \Pi_k$ implies that either world w does not have two π-children or it has two π-children w_1 and w_2 that belong to Δ_k. By the induction hypothesis, there are two complete embedded models $\langle M_1, h_1\rangle$ and $\langle M_2, h_2\rangle$ and worlds v_1 and v_2 of those models such that $h_1(v_1) = w_1$ and $h_2(v_2) = w_2$. Similarly, $w \in \Sigma_k$ implies that if w has a σ_i-child w_3 then there is a complete embedded model $\langle M_2, h_3\rangle$ and a world v_3 of M_3 such that $h_3(v_3) = w_3$. We can combine those, at most three, complete embedded models into one complete embedded model $\langle M, h\rangle$. Model M in addition to the worlds it inherits from models M_1, M_2, M_3 also has a new world v. Relations R_α are also inherited from models M_1, M_2, M_3 with extra elements (v, v_1), (v, v_2), and (v, v_3) added to R_{π_1},R_{π_2}, and R_{σ_i} correspondingly. Assume that values of $\|\cdot\|$ are also inherited from models M_1, M_2, M_3 and that $\|v\| = \|w\|$. Finally, let homomorphism h map v into w and is consistent with h_1,h_2, and h_3 on the other worlds. □

3.4 Canonical Structure

Definition 21 *A set of propositional formulas Φ is syntactically closed with respect to recursion ρ if*

(1) For any propositional variable p such that $\rho(p) \neq p$, both p and $\rho(p)$ belong to Φ.

(2) If $\phi_1 \to \phi_2 \in \Phi$, $\phi_1 \otimes \phi_2 \in \Phi$, or $\phi_1 \oplus \phi_2 \in \Phi$ then $\phi_1 \in \Phi$ and $\phi_2 \in \Phi$.
(3) $\phi \in \Phi$ if and only if $\bar{\phi} \in \Phi$ for any propositional formula ϕ.
(4) If $\phi_1 \otimes \phi_2 \in \Phi$ and $\psi_1 \otimes \psi_2 \in \Phi$ then $\bar{\phi}_1 \otimes \psi_2 \in \Phi$ and $\phi_1 \otimes \bar{\psi}_2 \in \Phi$.
(5) If $\phi_1 \oplus \phi_2 \in \Phi$ then $\bar{\phi}_1 \oplus \bar{\phi}_2 \in \Phi$.

Lemma 5 *Any finite set of propositional formulas could be extended to a finite syntactically closed set of formulas.* □

In the rest of this section we will assume that ρ is an arbitrary recursion, Φ is a finite syntactically closed with respect to ρ set of propositional formulas, $\mathbf{L}$ is one of two logics: $\mathbf{S}_\rho$ or $\mathbf{S}_\rho^\omega$. Let us define the canonical structure based on Φ and L:

Definition 22 *Let W be the set of all maximal $\mathbf{L}$-consistent subsets of Φ.*

Definition 23 *For any set $w \in W$ we define the following four projections: $pr_1^\pi(w) = \{\phi_1 \mid \exists\phi_2\ (\phi_1 \otimes \phi_2 \in w)\}$, $pr_2^\pi(w) = \{\phi_2 \mid \exists\phi_1\ (\phi_1 \otimes \phi_2 \in w)\}$, $pr_1^\sigma(w) = \{\phi_1 \mid \exists\phi_2\ (\phi_1 \oplus \phi_2 \in w)\}$, $pr_2^\sigma(w) = \{\phi_2 \mid \exists\phi_1\ (\phi_1 \oplus \phi_2 \in w)\}$.*

Definition 24 *For any two sets $w, u \in W$, any $\alpha \in \{\pi, \sigma\}$, and any index $i \in \{1, 2\}$, let $wR_{\alpha_i}u$ be true if and only if $pr_i^\alpha(w) \neq \varnothing$ and $pr_i^\alpha(w) \subseteq u$.*

Definition 25 *For any $w \in W$, let $\|w\| = \{p \in Var \mid p \in w\}$.*

Lemma 6 *$\langle W, R, L, \|\cdot\|\rangle$ is a Kripke structure.* □

The above defined Kripke structure will be called canonical structure of logic $\mathbf{L}$ based on the syntactically closed set of formulas Φ.

3.5 Properties of Canonical Structure

Lemma 7 *For any $w \in W$ and any formula $\phi \otimes \psi$, if $\phi \otimes \psi \in w$ then there are $u, v \in W$ such that $wR_1^\pi u$ and $wR_2^\pi v$.*

Proof. Since any $\mathbf{L}$-consistent set can be extended to a maximal $\mathbf{L}$-consistent set, it is sufficient to show that $pr_1^\pi(w)$ and $pr_2^\pi(w)$ are consistent. Let us first prove that $pr_1^\pi(w)$ is consistent. Indeed, let $\phi_1 \otimes \psi_1, \ldots, \phi_n \otimes \psi_n$ be the list of formulas in w whose outermost operation is $\otimes$. This list is not empty because $\phi \otimes \psi$ is on the list. Assume that $\mathbf{L} \vdash \phi_1, \ldots, \phi_n \Rightarrow$ and use the following instance of (XP) rule:

$$\frac{\phi_1, \ldots, \phi_n \Rightarrow}{\phi_1 \otimes \psi_1, \ldots, \phi_n \otimes \psi_n \Rightarrow}$$

to conclude that w is not consistent. Contradiction. Similarly, one can show that $pr_2^\pi(w)$ is also consistent. □

Lemma 8 *For any $w \in W$ and any formula $\phi \oplus \psi$, if $\phi \oplus \psi \in w$ then there is $u \in W$ such that either $wR^{\sigma}_1 u$ or $wR^{\sigma}_2 u$.*

Proof. Since any **L**-consistent set can be extended to a maximal **L**-consistent set, it is sufficient to show that either $pr^{\sigma}_1(w)$ or $pr^{\sigma}_2(w)$ is consistent. Indeed, let $\phi_1 \oplus \psi_1, \dots, \phi_n \oplus \psi_n$ be the list of formulas in w whose outermost operation is $\oplus$. This list is not empty because $\phi \oplus \psi$ is on the list. Assume that $\mathbf{L} \vdash \phi_1, \dots, \phi_n \Rightarrow$ and $\mathbf{L} \vdash \psi_1, \dots, \psi_n \Rightarrow$. By (XU) rule, $\mathbf{L} \vdash \phi_1 \oplus \psi_1, \dots, \phi_n \oplus \psi_n \Rightarrow$. Contradiction with the consistency of w. $\square$

Lemma 9 *For any $\phi_1 \to \phi_2 \in \Phi$ and any $w \in W$, $\phi_1 \to \phi_2 \in w$ if and only if either $\phi_1 \notin w$ or $\phi_2 \in w$.* $\square$

Lemma 10 *For any sets $w, v, u \in W$ such that $wR_{\pi_1}u$ and $wR_{\pi_2}v$ and any formula $\phi_1 \otimes \phi_2 \in \Phi$, $\phi_1 \otimes \phi_2 \in w$ if and only if $\phi_1 \in u$ and $\phi_2 \in v$.*

Proof. ($\Rightarrow$): By Definition 23, $\phi_1 \otimes \phi_2 \in w$ implies that $\phi_1 \in pr^{\pi}_1(w)$ and $\phi_2 \in pr^{\pi}_2(w)$. At the same time, $pr^{\pi}_1(w) \subseteq u$ and $pr^{\pi}_2(w) \subseteq v$. Thus, $\phi_1 \in u$ and $\phi_2 \in v$. ($\Leftarrow$): Assumption $wR_{\pi_1}u$ implies that $pr^{\pi}_1(w)$ is not empty. Thus, $\psi_1 \otimes \psi_2 \in w$ for some formula $\psi_1 \otimes \psi_2$ in Φ. Assume that $\phi_1 \otimes \phi_2 \notin w$. By Definition 21, $\neg(\phi_1 \otimes \phi_2) \in \Phi$. Since w is a maximal consistent subset of Φ, formula $\neg(\phi_1 \otimes \phi_2)$ should belong to w. By Lemma 1, $\mathbf{L} \vdash w \Rightarrow \bar{\phi}_1 \otimes \psi_2, \psi_1 \otimes \bar{\phi}_2$. Since, by Definition 21, both $\bar{\phi}_1 \otimes \psi_2$ and $\psi_1 \otimes \bar{\phi}_2$ belong to Φ, maximality of w implies that $\bar{\phi}_1 \otimes \psi_2 \in w$ or $\psi_1 \otimes \bar{\phi}_2 \in w$. Thus, $\bar{\phi}_1 \in pr^{\pi}_1(w) \subseteq u$ or $\bar{\phi}_2 \in pr^{\pi}_2(w) \subseteq v$. Consistency of u and v implies that $\phi_1 \notin u$ or $\phi_2 \notin v$. $\square$

Lemma 11 *For any $i \in \{1, 2\}$ and any sets $w, u \in W$ such that $wR_{\sigma_i}u$ and any formula $\phi_1 \oplus \phi_2 \in \Phi$, $\phi_1 \oplus \phi_2 \in w$ if and only if $\phi_i \in u$.*

Proof. Let us consider case $i = 1$, the other case is similar. ($\Rightarrow$): By Definition 23, $\phi_1 \oplus \phi_2 \in w$ implies that $\phi_1 \in pr^{\sigma}_1(w)$. At the same time, $pr^{\sigma}_1(w) \subseteq u$. Thus, $\phi_1 \in u$. ($\Leftarrow$): Assumption $wR_{\sigma_1}u$ implies that $pr^{\sigma}_1(w)$ is not empty. Thus, $\psi_1 \oplus \psi_2 \in w$ for some formula $\psi_1 \oplus \psi_2$ in Φ. Assume that $\phi_1 \oplus \phi_2 \notin w$. By Definition 21, $\neg(\phi_1 \oplus \phi_2) \in \Phi$. Since w is a maximal consistent subset of Φ, formula $\neg(\phi_1 \oplus \phi_2)$ should belong to w. By Lemma 2, $\mathbf{L} \vdash w \Rightarrow \bar{\phi}_1 \oplus \bar{\phi}_2$. By Definition 21, $\bar{\phi}_1 \oplus \bar{\phi}_2$ belongs to Φ. Thus $\bar{\phi}_1 \in pr^{\sigma}_1(w) \subseteq u$. Consistency of u implies that $\phi_1 \notin u$. $\square$

Lemma 12 *For any world w of an embedded model $\langle M, h\rangle$ of the canonical Kripke structure and any formula $\phi \in \Phi$, $w \Vdash \phi$ if and only if $\phi \in h(w)$.*

Proof. Induction on formula ϕ complexity. Atomic case is trivial. If ϕ is an implication, then the required follows from Lemma 9. Let us consider the remaining cases.

Assume that ϕ be a product $\phi_1 \otimes \phi_2$. ($\Rightarrow$): $w \Vdash \phi$ implies that w has π-children $\pi_1(w)$, $\pi_2(w)$ and $\pi_1(w) \Vdash \phi_1$, $\pi_2(w) \Vdash \phi_2$. Hence, by the induction hypothesis, $\phi_1 \in h(\pi_1(w))$ and $\phi_2 \in h(\pi_2(w))$. By Lemma 10, the last conjunction implies $\phi_1 \otimes \phi_2 \in h(w)$. ($\Leftarrow$): By Lemma 7, $\phi \in h(w)$ implies that $h(w)$ has π_1- and π_2-children. Since w is π-complete under embedding h, it also has π-children $\pi_1(w)$ and $\pi_2(w)$. Since h is a homomorphism, $h(w)R_1^{\pi}h(\pi_1(w))$ and $h(w)R_2^{\pi}h(\pi_2(w))$. Hence, by Lemma 10, $\phi \in h(w)$ implies that $\phi_1 \in h(\pi_1(w))$ and $\phi_2 \in h(\pi_2(w))$. By the induction hypothesis, $\pi_1(w) \Vdash \phi_1$ and $\pi_2(w) \Vdash \phi_2$. Therefore, $w \Vdash \phi$.

Suppose that ϕ be a disjoint union $\phi_1 \oplus \phi_2$. ($\Rightarrow$): $w \Vdash \phi$ implies that either w has a σ_1-child and $\sigma_1(w) \Vdash \phi_1$ or w has a σ_2-child and $\sigma_2(w) \Vdash \phi_2$. Let us consider the first case. By the induction hypothesis, $\phi_1 \in h(\sigma_1(w))$. Thus, by Lemma 11, $\phi_1 \oplus \phi_2 \in h(w)$. The second case is similar. ($\Leftarrow$): By Lemma 8, $\phi \in h(w)$ implies that $h(w)$ has either σ_1- or σ_2-child. Let us consider the first case. Since w is σ-complete under embedding h, it also has σ_1-child $\sigma_1(w)$. Since h is a homomorphism, $h(w)R_1^{\sigma}h(\sigma_1(w))$. Hence, by Lemma 11, $\phi \in h(w)$ implies that $\phi_1 \in h(\sigma_1(w))$. By the induction hypothesis, $\sigma_1(w) \Vdash \phi_1$. Therefore, $w \Vdash \phi$. The second case is similar. $\Box$

Next, we will prove the completeness ($\Leftarrow$) part of Theorem 1.

3.6 Kripke Completeness of Logic $\mathbf{S}_\rho$

Let $\mathbf{S}_\rho \nvdash \Delta \Rightarrow \Gamma$. Thus, $X = \Delta \cup \{\neg\gamma \mid \gamma \in \Gamma\}$ is $\mathbf{S}_\rho$-consistent. By Lemma 5, set X can be extended to a finite syntactically complete set Φ. Consider canonical Kripke structure based on Φ. Let u be a maximal $\mathbf{S}_\rho$-consistent subset of Φ containing X. By Lemma 3, there is node w of an embedded Kripke model $\langle M, h\rangle$ such that $h(w) = v$. By Lemma 12, $w \Vdash \phi$ for any $\phi \in v$. Thus, $w \Vdash \phi$ for any $\phi \in X$.

Let us show that model M is ρ-sound. Indeed, if $\rho(p) = p$ then, $u \Vdash \rho(p)$ if and only if $u \Vdash p$ for any $u \in W$. Assume $\rho(p) \neq p$. Hence, by Definition 21, both p and $\rho(p)$ belong to Φ. Thus, by Lemma 12, it will be sufficient to establish that $p \in u$ if and only if $\rho(p) \in u$ for any $u \in W$. ($\Rightarrow$): Suppose $p \in u$. By (RR) rule, $\mathbf{S}_\rho \vdash u \Rightarrow \rho(p)$. Since $\rho(p) \in \Phi$ and u is a maximal $\mathbf{S}_\rho$-consistent subset of Φ, formula $\rho(p)$ must belong to u. ($\Leftarrow$): If $p \notin u$ then, by maximality of u, we have $s_\rho \vdash u, p \Rightarrow$. Hence, by (LR) rule, $s_\rho \vdash u, \rho(p) \Rightarrow$. Since u is $\mathbf{S}_\rho$-consistent, $\rho(p) \notin u$. $\Box$

Definition 26 *For any Kripke model $K = \langle W, R, L, \|\cdot\|\rangle$, any $u \in W$, and any $n \geq 0$, let K_u^n be a restriction of this model to worlds from $W_u^n = \{v \mid u = v_0Rv_1R\dots Rv_k = v, 0 \leq k \leq n\}$.*

Lemma 13 *For any world u of an arbitrary Kripke model K, any $n \geq 0$ and an arbitrary modal propositional formula ϕ that has no more than n instances of connectives, if $\Vdash$ refers to forcing relation on model K and $\Vdash'$ to the forcing relation on model K_u^n then $w \Vdash \phi$ if and only if $w \Vdash' \phi$.*

Proof. Induction on n. $\square$

Back to the completeness proof. Suppose $\mathbf{S} \nvdash \Delta \Rightarrow \Gamma$. Let n be the maximal number of connectives among formulas in Γ and Δ. Since $\mathbf{S} = \mathbf{S}_{id}$, completeness of $\mathbf{S}_\rho$ implies that there is a world u of a Kripke model W such that $u \Vdash \gamma$ for all $\gamma \in \Gamma$ and $u \nVdash \delta$ for all $\delta \in \Delta$. Consider model W_u^n. It is finite and, by Lemma 13, in this model $u \Vdash \gamma$ for all $\gamma \in \Gamma$ and $u \nVdash \delta$ for all $\delta \in \Delta$. Therefore, $\nVdash_{id}^F \Gamma \Rightarrow \Delta$. $\square$

3.8 Kripke Completeness of Logic $\mathbf{S}_\rho^\omega$

Proof of completeness for logic $\mathbf{S}_\rho^\omega$ is similar to the one for logic $\mathbf{S}_\rho$, but instead of Lemma 3 one should use Lemma 4. Thus, we only need to show that in the case of logic $\mathbf{S}_\rho^\omega$, the canonical Kripke structure defined above is well-founded.

Definition 27 *For any finite set of propositional formulas v, by $\wedge v$ we mean conjunction of all formulas in v. For any finite set V of finite sets of propositional formulas, by $\bigvee V$ we mean disjunction of $\wedge v$ for all $v \in V$.*

Lemma 14 *For any $w \in W$ such that $pr_1^\pi(w)$ and $pr_2^\pi(w)$ are not empty,*

$$\mathbf{S}_\rho^\omega \vdash w \Rightarrow (\wedge(pr_1^\pi(w))) \otimes (\wedge(pr_2^\pi(w))).$$

Proof. Let $\phi_1 \otimes \psi_1, \dots, \phi_n \otimes \psi_n$ be the list of all formulas in w whose outer-most operation is $\otimes$. We need to establish that $\mathbf{S}_\rho^\omega \vdash w \Rightarrow (\bigwedge_i \phi_i) \otimes (\bigwedge_i \psi_i)$. Indeed, the following two sequents are provable in classical logic: $\phi_1, \dots, \phi_n \Rightarrow \bigwedge_i \phi_i$ and $\psi_1, \dots, \psi_n \Rightarrow \bigwedge_i \psi_i$. By combining them with (XP) rule, we get

$$\mathbf{S}_\rho^\omega \vdash \phi_1 \otimes \psi_1, \dots, \phi_n \otimes \psi_n \Rightarrow (\bigwedge_i \phi_i) \otimes (\bigwedge_i \psi_i).$$

Finally, multiple applications of (LW) rule to the above sequent result in: $\mathbf{S}_\rho^\omega \vdash w \Rightarrow (\bigwedge_i \phi_i) \otimes (\bigwedge_i \psi_i)$. $\square$

Lemma 15 *For any $w \in W$ such that $pr_1^\sigma(w)$ and $pr_2^\sigma(w)$ are not empty,*

$$\mathbf{S}_\rho^\omega \vdash w \Rightarrow (\wedge(pr_1^\sigma(w))) \oplus (\wedge(pr_2^\sigma(w))).$$

Proof. Similar to Lemma 14, but use (XU) rule instead of the (XP) rule. □

Lemma 16 *For any $X \subseteq \Phi$, $\mathbf{S}_\rho^\omega \vdash \wedge X \Rightarrow \bigvee_{X \subseteq u \in W}(\wedge u)$.*

Proof. Assume the opposite. Thus, set $X \cup \{\neg(\wedge u) \mid X \subseteq u \in W\}$ is $\mathbf{S}_\rho^\omega$-consistent. Let α be a maximal (infinite) $\mathbf{S}_\rho^\omega$-consistent extension of this set. Since α is $\mathbf{S}_\rho^\omega$-consistent,

$$\forall u \in W (X \subseteq u \Rightarrow \exists \phi \in u\ (\neg\phi \in \alpha)) \tag{1}$$

Let $u_\alpha = \Phi \cap \alpha$. Note that u_α is a maximal $\mathbf{S}_\rho^\omega$-consistent subset of Φ. Hence, $u_\alpha \in W$. Since X is a subset of Φ and α, we can claim that $X \subseteq u_\alpha$. Thus, according to (1), there is $\phi \in u_\alpha$ such that $\neg\phi \in \alpha$. Since $u_\alpha \subseteq \alpha$, set α contains both ϕ and $\neg\phi$. Therefore, α is not $\mathbf{S}_\rho^\omega$-consistent. Contradiction. □

Lemma 17 *For any $w \notin \Pi_0$, $\mathbf{S}_\rho^\omega \vdash w \Rightarrow (\bigvee_{wR_{\pi_1}u}(\wedge u)) \otimes (\bigvee_{wR_{\pi_2}v}(\wedge v))$.*

Proof. Note that since $w \notin \Pi_0$, sets $pr_1^\pi(w)$ and $pr_1^\pi(w)$ are not empty. Consider Lemma 16 for two cases: $X = pr_1^\pi(w)$ and $X = pr_2^\pi(w)$:

$$\mathbf{S}_\rho^\omega \vdash \wedge(pr_1^\pi(w)) \Rightarrow \bigvee_{wR_{\pi_1}u}(\wedge u), \qquad \mathbf{S}_\rho^\omega \vdash \wedge(pr_2^\pi(w)) \Rightarrow \bigvee_{wR_{\pi_2}u}(\wedge u).$$

We can use (XP) rule to combine the last two statements into one:

$$\mathbf{S}_\rho^\rho \vdash (\wedge(pr_1^\pi(w))) \otimes (\wedge(pr_2^\pi(w))) \Rightarrow (\bigvee_{wR_{\pi_1}u}(\wedge u)) \otimes (\bigvee_{wR_{\pi_2}u}(\wedge u)).$$

The last statement, when combined with Lemma 14, implies that

$$\mathbf{S}_\rho^\omega \vdash w \Rightarrow (\bigvee_{wR_{\pi_1}u}(\wedge u)) \otimes (\bigvee_{wR_{\pi_2}v}(\wedge v)). \qquad \Box$$

Lemma 18 *$\mathbf{S}_\rho^\omega \vdash w \Rightarrow (\bigvee_{wR_{\sigma_1}u}(\wedge u)) \oplus (\bigvee_{wR_{\sigma_2}v}(\wedge v))$, for any $w \notin \Sigma_0$.*

Proof. Assumption $w \notin \Sigma_0$ implies that one of $pr_1^\sigma(w), pr_2^\sigma(w)$ is not empty. Thus, there are ϕ and ψ such that $\phi \oplus \psi \in w$. Hence, both $pr_1^\sigma(w)$ and $pr_2^\sigma(w)$ are not empty. From here proceed as in Lemma 17, but use (XU) rule instead of (XP) and Lemma 15 instead of Lemma 14. □

Lemma 19 *$\mathbf{S}_\rho^\omega \vdash w \Rightarrow (\bigvee\Omega) \otimes (\bigvee W), (\bigvee W) \otimes (\bigvee\Omega)$, for any $w \notin \bigcup_i \Pi_i$.*

Proof. By Definition 12, assumption $w \notin \bigcup_i \Pi_i$ implies either $\{u \mid wR_{\pi_1}u\} \subseteq \Omega$ or $\{u \mid wR_{\pi_2}u\} \subseteq \Omega$. Let us start with the first case. Since $\{u \mid wR_{\pi_1}u\} \subseteq \Omega$, the following sequent is trivially provable in the classical logic: $\bigvee_{wR_{\pi_1}u}(\wedge u) \Rightarrow \bigvee \Omega$. Similarly, sequent $\bigvee_{wR_{\pi_1}u}(\wedge u) \Rightarrow \bigvee W$ is provable in the classical logic because $\{u \,|\, wR_2^{\sigma}u\}$ is a subset of W. By applying (XP) rule to these two sequents, we get $\mathbf{S}^{\omega}_{\rho} \vdash (\bigvee_{wR_{\pi_1}u}(\wedge u)) \otimes (\bigvee_{wR_{\pi_1}u}(\wedge u)) \Rightarrow (\bigvee \Omega) \otimes (\bigvee W)$. The last statement, when combined with Lemma 17 implies that $\mathbf{S}^{\omega}_{\rho} \vdash w \Rightarrow (\bigvee \Omega) \otimes (\bigvee W)$. Finally, by (RW) rule, $\mathbf{S}^{\omega}_{\rho} \vdash w \Rightarrow (\bigvee \Omega) \otimes (\bigvee W), (\bigvee W) \otimes (\bigvee \Omega)$. The case $\{u \,|\, wR_{\pi_2}u\} \subseteq \Omega$ is similar. □

Lemma 20 $\mathbf{S}^{\omega}_{\rho} \vdash w \Rightarrow (\bigvee \Omega) \oplus (\bigvee \Omega)$, *for any* $w \notin \bigcup_i \Sigma_i$.

Proof. By Definition 12, assumption $w \notin \bigcup_i \Sigma_i$ implies that $\{u \,|\, wR_{\sigma_1}u\} \subseteq \Omega$ and $\{u \,|\, wR_{\sigma_2}u\} \subseteq \Omega$. Thus, the following two sequents are provable in the classical logic: $\bigvee_{wR_{\sigma_1}u}(\wedge u) \Rightarrow \bigvee \Omega$ and $\bigvee_{wR_{\sigma_1}u}(\wedge u) \Rightarrow \bigvee \Omega$. By (XU) rule, $\mathbf{S}^{\omega}_{\rho} \vdash (\bigvee_{wR_{\sigma_1}u}(\wedge u)) \oplus (\bigvee_{wR_{\sigma_1}u}(\wedge u)) \Rightarrow (\bigvee \Omega) \oplus (\bigvee \Omega)$. The last statement, when combined with Lemma 18 implies that $\mathbf{S}^{\omega}_{\rho} \vdash w \Rightarrow (\bigvee \Omega) \oplus (\bigvee \Omega)$. □

Lemma 21 *Kripke structure* $K = \langle W, R, L, \|\cdot\|\rangle$ *is well-founded.*

Proof. Since $\Delta = (\bigcup_i \Pi_i) \cap (\bigcup_i \Sigma_i)$, any element of Ω can not belong to both: $\bigcup_i \Pi_i$ and $\bigcup_i \Sigma_i$. Thus, according to Lemma 19 and Lemma 20, for any $w \in \Omega$ $\mathbf{S}^{\omega}_{\rho} \vdash w \Rightarrow (\bigvee \Omega) \otimes (\bigvee W), (\bigvee W) \otimes (\bigvee \Omega)$ or $\mathbf{S}^{\omega}_{\rho} \vdash w \Rightarrow (\bigvee \Omega) \oplus (\bigvee \Omega)$. Therefore, by (RW) rule, $\mathbf{S}^{\omega}_{\rho} \vdash w \Rightarrow (\bigvee \Omega) \otimes (\bigvee W), (\bigvee W) \otimes (\bigvee \Omega), (\bigvee \Omega) \oplus (\bigvee \Omega)$, for any $w \in \Omega$. Hence, $\mathbf{S}^{\omega}_{\rho} \vdash \wedge w \Rightarrow (\bigvee \Omega) \otimes (\bigvee W), (\bigvee W) \otimes (\bigvee \Omega), (\bigvee \Omega) \oplus (\bigvee \Omega)$, for any $w \in \Omega$. Thus, $\mathbf{S}^{\omega}_{\rho} \vdash \bigvee \Omega \Rightarrow (\bigvee \Omega) \otimes (\bigvee W), (\bigvee W) \otimes (\bigvee \Omega), (\bigvee \Omega) \oplus (\bigvee \Omega)$. By (WF) rule, $\mathbf{S}^{\omega}_{\rho} \vdash \bigvee \Omega \Rightarrow$. This implies that $\mathbf{S}^{\omega}_{\rho} \vdash w \Rightarrow$ for all $w \in \Omega$. This means that every element of Ω is inconsistent. Therefore, Ω is empty. □

This concludes Kripke completeness proof for logic $\mathbf{S}^{\omega}_{\rho}$.

4 Term Semantics

Definition 28 *Let* $\langle W, R, L, \|\cdot\|\rangle$ *be an arbitrary Kripke model. We say that pair* $\langle w_1, w_2\rangle$ *of worlds in this model is* π*-grounded if there is a world* v *such that* $vR_1^{\pi}w_1$ *and* $vR_2^{\pi}w_2$*.*

Suppose two worlds w_1 and w_2 of some Kripke model $\langle W, R, L, \|\cdot\|\rangle$ are not grounded. They always can be made grounded by an introduction of a new element w into set W and pairs $\langle w, w_1\rangle$ and $\langle w, w_2\rangle$ to relations R_1^{π} and R_2^{π} correspondingly. Value of $\|w\|$ could be assigned arbitrarily. It is clear that such extension of a Kripke model does not change relation $u \Vdash \phi$ for any propositional formula ϕ and any world u as long as $u \neq w$.

Lemma 22 *For any positive recursion* ρ*, if the above construction starts with a* ρ*-sound Kripke model then value* $\|w\|$ *could be defined in such a way that* *the resulting Kripke model is also ρ-sound.*

Proof. Consider predicate transformer $\tau : 2^{Var} \mapsto 2^{Var}$ that is defined as follows: For any $X \subseteq Var$ define $\|w\|$ to be X. Let $\tau(X) = \{p \in Var \| w \Vdash \rho(p)$. It is easy to see that because recursion ρ is positive, predicate transformer τ is monotonic. Thus, it has a fixed point X_0. If $\|w\|$ is defined to be X_0 then the model is ρ-sound. $\square$

Definition 29 *Let $\langle W, R, L, \|\cdot\|\rangle$ be an arbitrary Kripke model. We say that world w in this model is σ-grounded if there are worlds u and v such that $uR_1^\sigma w$ and $vR_2^\sigma w$.*

Using an argument similar to the one above, it can be shown that any world of a ρ-sound Kripke model can be made σ-grounded by an introduction of at most two new worlds to the model. Such model modification will preserve $\Vdash$ relation on existing worlds and, using predicate transformer technique from Lemma 22, the new Kripke model can be made ρ-sound.

Definition 30 *A Kripke model is grounded if every world of this model is π-grounded and σ-grounded at the same time.*

Lemma 23 *For any positive recursion ρ, every ρ-sound Kripke model can be extended to a ρ-sound grounded Kripke model preserving $\Vdash$ relation on the existing worlds.*

Proof. Use described above construction to add new worlds to the model that will make all original worlds π- and σ-grounded. Repeat this process infinitely many times to obtain an expanding chain of Kripke models. Every world will be π- and σ-grounded starting with some element of the chain. Take the union of the chain to be the final Kripke model. $\square$

Definition 31 *A Kripke model $\langle W, R, L, \|\cdot\|\rangle$ is well-founded if any chain $w_0Rw_1R\dots Rw_nR\dots$ is finite.*

Lemma 24 *For any positive recursion ρ, every finite ρ-sound Kripke model can be extended to a ρ-sound well-founded grounded Kripke model preserving $\Vdash$ relation on the existing worlds.*

Proof. The algorithm described in the proof of Lemma 23, when applied to a final Kripke model, produces a well-founded model. $\square$

Definition 32 *For any grounded Kripke model $K = \langle W, R, L, \|\cdot\|\rangle$ the universe $U_K = \langle Term, pair, inl, inr\rangle$ is defined as follows: $Term = W$, $pair(v, w)$ is a world u of the grounded model K such that $uR_1^\pi v$ and $uR_2^\pi w$, $inl(v)$ is a world u of the grounded model K such that $uR_1^\sigma v$, $inr(v)$ is a world u of the grounded model K such that $uR_2^\sigma v$.*

Lemma 25 *For any well-founded grounded Kripke model K, universe U_K is also well-founded.* □

Definition 33 *For any grounded Kripke model K, valuation $*_K$ over the universe U_K is defined as follows: $p^{*_K} = \{w \mid w \Vdash p\}$.*

Lemma 26 *For any grounded Kripke model $K = \langle W, R, L, \|\cdot\|\rangle$, any $w \in W$, and any propositional formula ϕ, $w \Vdash \phi$ if and only if $w \in \phi^{*_K}$.*

Proof. Induction on the structural complexity of ϕ. □

Corollary 3 *For any ρ-sound grounded Kripke model K, valuation $*_K$ is also ρ-sound.* □

We are finally ready to finish proofs of Theorem 1. Suppose $\mathbf{S}_\rho \nvdash \Gamma \Rightarrow \Delta$. By Theorem 2, there is a Kripke model $\langle W, R, L, \|\cdot\|\rangle$ and a world $w \in W$ such that $w \Vdash \gamma$ for all $\gamma \in \Gamma$ and $w \nVdash \delta$ for all $\delta \in \Delta$. By Lemma 23, this model could be extended to a grounded well-founded Kripke model K. Consider universe U_K. By Lemma 1, $w \in \gamma^{*_K}$ for all $\gamma \in \Gamma$ and $w \notin \delta^{*_K}$ for all $\delta \in \Delta$. Hence, $\bigcap \Gamma^{*_K} \nsubseteq \bigcup \Delta^{*_K}$. Thus, $\nVdash_\rho \Gamma \Rightarrow \Delta$.

Cases of logic $\mathbf{S}$ and $\mathbf{S}_\rho^\omega$ are similar to the one above, but use Lemma 24 instead of Lemma 23. □

5 Cut Elimination in S

In this section we will prove Cut elimination theorem for $\mathbf{S}$ following Cut elimination proof for classical logic in Troelstra and Schwichtenberg [2000].

5.1 Absorption of Structural Rules

Definition 34 *Cut-free Logic of Subtyping with Absorbed structural rules A is defined by inference rules listed on Figure 1.*

We will refer to (XP) and (XU) rules as X-rules. A formula in the conclusion of an X-rule is called *principal*, if it does not belong to Γ or Δ. Principal formula of other inference rules is defined as usual. Let A^+ be the extension of A by Cut rule and $\mathbf{S}^-$ be Cut-free fragment of $\mathbf{S}$.

Theorem 3 *For any two multisets of propositional formulas Γ and Δ, $A^+ \vdash \Gamma \Rightarrow \Delta$ iff $\mathbf{S} \vdash \Gamma \Rightarrow \Delta$ and $A \vdash \Gamma \Rightarrow \Delta$ iff $\mathbf{S}^- \vdash \Gamma \Rightarrow \Delta$.* □

$$\Gamma, p \Rightarrow \Delta, p \ (\textsc{ax}) \qquad \Gamma, \bot \Rightarrow \Delta \ (\textsc{lf})$$

$$\frac{\Gamma \Rightarrow \phi, \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \to \psi \Rightarrow \Delta} (\textsc{li}) \qquad \frac{\Gamma, \phi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \to \psi, \Delta} (\textsc{ri})$$

$$\frac{\forall J' \subseteq J(\phi_I \Rightarrow \psi_{J'} \text{ or } \chi_I \Rightarrow \eta_{J \setminus J'})}{\Gamma, \{\phi_i \otimes \chi_i\}_{i \in I} \Rightarrow \Delta, \{\psi_j \otimes \eta_j\}_{i \in J}} (\textsc{xp})$$

$$\frac{\phi_I \Rightarrow \psi_J \qquad \chi_I \Rightarrow \eta_J}{\Gamma, \{\phi_i \oplus \chi_i\}_{i \in I} \Rightarrow \Delta, \{\psi_j \oplus \eta_j\}_{i \in J}} (\textsc{xu})$$

Fig. 1. Logic A. It is assumed that $|I| > 0$.

Let $A \vdash^d \phi$ mean that ϕ has an A derivation of depth no more than d.

Lemma 27 *For any $d \geq 0$ and any multisets of formulas Γ, Δ, Γ', and Δ', if $A \vdash^d \Gamma \Rightarrow \Delta$, then $A \vdash^d \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$.*

Proof. Induction on d. Axioms and inference rules (RI) and (LI) can be handled the same way as in the case of the classical propositional logic. Suppose $\Gamma \Rightarrow \Delta$ is deducted by (XU) rule. It means that $\Gamma \equiv \Gamma_0, \{\phi_i \oplus \chi_i\}_{i \in I}$ and $\Delta \equiv \Delta_0, \{\psi_j \oplus \eta_j\}_{j \in J}$ for some multisets $\Gamma_0, \Delta_0, \{\phi_i \oplus \chi_i\}_{i \in I}, \{\psi_j \oplus \eta_j\}_{j \in J}$, such that $|I| > 0$, $A \vdash^{d-1} \phi_I \Rightarrow \psi_J$, and $A \vdash^{d-1} \chi_I \Rightarrow \eta_J$. Applying rule (XU), we get $A \vdash^d \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$. (XP) rule can be handled similarly. $\Box$

The proof of the above lemma provides an algorithm to convert any derivation $\mathcal{D}$ of $\Gamma \Rightarrow \Delta$ into a derivation of $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ of the same depth. We will denote the resulting derivation as $\mathcal{D}[\Gamma' \Rightarrow \Delta']$.

Corollary 4 *Weakening rule* $\dfrac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$ (W) *is admissible in A.* $\Box$

Lemma 28 *For any $d \geq 0$ and any multisets of propositional formulas Γ, Γ', Δ, and Δ', if $\vdash^d_{A+} \Gamma \Rightarrow \Delta$, then $\vdash^d_{A+} \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$.*

Proof. Induction on the size of proof of $\Gamma \Rightarrow \Delta$. Similar to the proof of Lemma 27 with addition of one more case corresponding to (C) rule. $\Box$

Lemma 29 *For any propositional formula ϕ, $A \vdash \phi \Rightarrow \phi$.*

Proof. Induction on the size of ϕ. Cases of ϕ being a propositional variable or an implication are identical to those for the classical propositional logic.

Assume that $\phi \equiv \chi \otimes \psi$. Since by the induction hypothesis $\mathbf{S}_{id} \vdash \chi \Rightarrow \chi$ and $\mathbf{S}_{id} \vdash \psi \Rightarrow \psi$, we can use the following instance of (XP) rule

$$\frac{\chi \Rightarrow \chi \qquad \psi \Rightarrow \psi}{\chi \otimes \psi \Rightarrow \chi \otimes \psi} \tag{2}$$

to conclude that $A \vdash \chi \otimes \psi \Rightarrow \chi \otimes \psi$. In the formula (2), J is a single-element set. Thus, there are only two subsets of J: set J itself and empty set $\varnothing$. The first premise of (2) corresponds to subset J and the second premise corresponds to empty subset.

Suppose $\phi \equiv \chi \oplus \psi$. By the induction hypothesis $A \vdash \chi \Rightarrow \chi$ and $A \vdash \psi \Rightarrow \psi$. We can apply (XU) rule to conclude that $A \vdash \chi \oplus \psi \Rightarrow \chi \oplus \psi$. $\square$

Lemma 30 *For any $d \geq 0$, any multisets of propositional formulas Γ and Δ, and any propositional formulas ϕ and ψ, if $A \vdash^d \Gamma, \phi \to \psi \Rightarrow \Delta$ then $A \vdash^d \Gamma \Rightarrow \Delta, \phi$ and $A \vdash^d \Gamma, \psi \Rightarrow \Delta$.*

Proof. Induction on d. Axioms and inference rules (RI) and (LI) can be handled the same way as in the case of the classical propositional logic Assume that $\Gamma, \phi \to \psi \Rightarrow \Delta$ is derived by (XU) rule. Thus, the last step of the derivation has the following form

$$\frac{\phi_I \Rightarrow \psi_J \qquad \chi_I \Rightarrow \eta_J}{\Gamma, \phi \to \psi, \{\phi_i \oplus \chi_i\}_{i \in I} \Rightarrow \Delta, \{\psi_j \oplus \eta_j\}_{j \in J}} \text{(XU)}.$$

One can use derivations of $\phi_I \Rightarrow \psi_J$ and $\chi_I \Rightarrow \eta_J$ and the same (XU) rule to derive in A sequents $\Gamma, \{\phi_i \oplus \chi_i\}_{i \in I} \Rightarrow \Delta, \phi\{\psi_j \oplus \eta_j\}_{j \in J}$ and $\Gamma, \psi, \{\phi_i \oplus \chi_i\}_{i \in I} \Rightarrow \Delta, \{\psi_j \oplus \eta_j\}_{j \in J}$. (XP) rule can be treated similarly. $\square$

Lemma 31 *For any $d \geq 0$, any multisets of propositional formulas Γ and Δ, and any propositional formulas ϕ and ψ, if $A \vdash^d \Gamma \Rightarrow \phi \to \psi, \Delta$ then $A \vdash^d \Gamma, \phi \Rightarrow \psi, \Delta$.*

Proof. Similar to the proof of Lemma 30. $\square$

Lemma 32 *A is closed under the following two contraction rules*

$$\frac{\Gamma, \alpha, \alpha \Rightarrow \Delta}{\Gamma, \alpha \Rightarrow \Delta}\text{(LC)} \qquad\qquad \frac{\Gamma \Rightarrow \alpha, \alpha, \Delta}{\Gamma \Rightarrow \alpha, \Delta}\text{(RC)}.$$

Proof. Induction on the depth d of the A derivation. Assume that $\Gamma, \alpha, \alpha \Rightarrow \Delta$ is an axiom. Then either there is a propositional variable p such that $p \in (\Gamma \cup \{\alpha\}) \cap \Delta$ or $\bot \in (\Gamma \cup \{\alpha\})$. In either of those cases $\Gamma, \alpha \Rightarrow \Delta$ is also an axiom. Rule (RC) can be handled similarly.

Suppose that $\Gamma, \alpha, \alpha \Rightarrow \Delta$ is derived by (LI) rule. If α is the principal formula in this rule then $\alpha \equiv \alpha_1 \to \alpha_2$ for some propositional formulas α_1 and α_2, $A \vdash^{d-1} \Gamma, \alpha_1 \to \alpha_2 \Rightarrow \alpha_1, \Delta$ and $A \vdash^{d-1} \Gamma, \alpha_1 \to \alpha_2, \alpha_2 \Rightarrow \Delta$. By Lemma 30, we can conclude that $A \vdash^{d-1} \Gamma \Rightarrow \alpha_1, \alpha_1, \Delta$ and $A \vdash^{d-1} \Gamma, \alpha_2, \alpha_2 \Rightarrow \Delta$. Thus, by the induction hypothesis, $A \vdash \Gamma \Rightarrow \alpha_1, \Delta$ and $A \vdash \Gamma, \alpha_2 \Rightarrow \Delta$. Finally, by (LI) rule, we get $A \vdash \Gamma, \alpha_1 \to \alpha_2 \Rightarrow \Delta$. If not α, but some other formula

$\sigma \to \tau$ is the principal formula of (LI) rule, then $A \vdash^{d-1} \Gamma, \alpha, \alpha \Rightarrow \sigma, \Delta$ and $A \vdash^{d-1} \Gamma, \alpha, \alpha, \tau \Rightarrow \Delta$. By the induction hypothesis, $A \vdash \Gamma, \alpha \Rightarrow \sigma, \Delta$ and $A \vdash \Gamma, \alpha, \tau \Rightarrow \Delta$. Hence, by (LI) rule, we can conclude that $A \vdash \Gamma, \alpha, \sigma \to \tau \Rightarrow \Delta$. Inference rule (RI) as well as the other contraction rule could be handled similarly.

Suppose that $\Gamma, \alpha, \alpha \Rightarrow \Delta$ is derived by (XU) rule. If at least one of two αs is not principal in this rule then the same rule could be used to derive the sequent without this instance of α. Assume that both αs are principal. Hence, $\alpha \equiv \alpha_1 \oplus \alpha_2$ for some formulas α_1 and α_2 and the instance of (XU) rule has the following form:

$$\frac{\alpha_1, \alpha_1, \phi_I \Rightarrow \psi_J \qquad \alpha_2, \alpha_2, \chi_I \Rightarrow \eta_J}{\Gamma_0, \alpha_1 \oplus \alpha_2, \alpha_1 \oplus \alpha_2, \{\phi_i \oplus \chi_i\}_{i \in I} \Rightarrow \Delta_0, \{\psi_j \oplus \eta_j\}_{j \in J}} \text{ (XU)}$$

By the induction hypothesis, $A \vdash \alpha_1, \phi_I \Rightarrow \psi_J$ and $A \vdash \alpha_2, \chi_I \Rightarrow \eta_J$. Applying (XU) rule, we get $A \vdash \Gamma_0, \alpha_1 \oplus \alpha_2, \{\phi_i \oplus \chi_i\}_{i \in I} \Rightarrow \Delta_0, \{\psi_j \oplus \eta_j\}_{j \in J}$. The other contraction rule could be handled similarly.

If $\Gamma, \alpha, \alpha \Rightarrow \Delta$ is derived by (XP) rule then we can show that $\Gamma, \alpha \Rightarrow \Delta$ using the same argument as in the previous case. Assume that $\Gamma \Rightarrow \alpha, \alpha, \Delta$ is derived by (XP) rule. We will show that $A \vdash \Gamma \Rightarrow \alpha, \Delta$. Indeed, if at least one of two αs is not principal in (XP) rule then the same rule could be used to derive the sequent without this instance of α. If both αs are principal then $\alpha \equiv \alpha_1 \oplus \alpha_2$ for some formulas α_1 and α_2, the conclusion of (XP) rule has the following form:

$$\Gamma_0, \{\phi_i \otimes \chi_i\}_{i \in I} \Rightarrow \Delta_0, \{\psi_j \otimes \eta_j\}_{j \in J}, \alpha_1 \otimes \alpha_2, \alpha_1 \otimes \alpha_2$$

and multiple premises of that rule could be arranged into the following three groups

(1) $\forall J' \subseteq J(\phi_I \Rightarrow \psi_{J'} \quad \text{or} \quad \chi_I \Rightarrow \eta_{I \setminus J'}, \alpha_2, \alpha_2)$,
(2) $\forall J' \subseteq J(\phi_I \Rightarrow \psi_{J'}, \alpha_1 \ \text{ or } \ \chi_I \Rightarrow \eta_{I \setminus J'}, \alpha_2)$,
(3) $\forall J' \subseteq J(\phi_I \Rightarrow \psi_{J'}, \alpha_1, \alpha_1 \quad \text{or} \quad \chi_I \Rightarrow \eta_{I \setminus J'})$.

By the induction hypothesis, from the first and the third of the above statements we can conclude that

$$\forall J' \subseteq J(A \vdash \phi_I \Rightarrow \psi_{J'} \text{ or } A \vdash \chi_I \Rightarrow \eta_{I \setminus J'}, \alpha_2),$$

$$\forall J' \subseteq J(A \vdash \phi_I \Rightarrow \psi_{J'}, \alpha_1 \text{ or } A \vdash \chi_I \Rightarrow \eta_{I \setminus J'}).$$

Therefore, by (XP) rule, $A \vdash \Gamma_0, \{\phi_i \otimes \chi_i\}_{i \in I} \Rightarrow \Delta_0, \{\psi_j \otimes \eta_j\}_{j \in J}, \alpha_1 \otimes \alpha_2$. $\square$

Theorem 4 *For any two multisets of modal propositional formulas Γ and Δ,*

$$A^{+} \vdash \Gamma \Rightarrow \Delta \quad \Longrightarrow \quad A \vdash \Gamma \Rightarrow \Delta$$

Proof. Consider any instance of the Cut rule in the derivation

$$\dfrac{\dfrac{\mathcal{D}_1}{\Gamma \Rightarrow \Delta, \alpha}\,(\textsc{r}1) \qquad \dfrac{\mathcal{D}_2}{\alpha, \Gamma \Rightarrow \Delta}\,(\textsc{r}2)}{\Gamma \Rightarrow \Delta}\,(\textsc{c}). \tag{3}$$

Let d be the depth of the derivation and s be the size of α. We will show by induction on $\langle s, d\rangle$ that if the left and the right subtrees of derivation (3) are cut-free then $A \vdash \Gamma \Rightarrow \Delta$. Just like in the Classical Logic case, there are three major cases to consider:

A. at least one of ($\textsc{r}1$) and ($\textsc{r}2$) is an axiom instance;
B. ($\textsc{r}1$) and ($\textsc{r}2$) are not axioms and at least in one of the premises α is not principal;
C. ($\textsc{r}1$) and ($\textsc{r}2$) are not axioms and α is principal in both premises.

Case A. Suppose that $\Gamma \Rightarrow \Delta, \alpha$ is an axiom. Then either (i) $\bot \in \Gamma$, or (ii) $\Gamma \cap \Delta$ contains a propositional variable, or (iii) α is a propositional variable and $\alpha \in \Gamma$. In the first two cases $\Gamma \Rightarrow \Delta$ is also an axiom. In the third case, we can apply ($\textsc{lc}$) rule (see Lemma 32) to sequent $\alpha, \Gamma \Rightarrow \Delta$ to conclude that $A \vdash \Gamma \Rightarrow \Delta$.

Case B. Although α could be principal in ($\textsc{r}1$) or ($\textsc{r}2$), we only will consider here the first of those subcases. Proof for the other one is similar. The first subcase will be split further based on what kind of rule ($\textsc{r}1$) turns out to be.

Let($\textsc{r}1$) be ($\textsc{li}$) rule. Then derivation (3) takes the form

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_{11}}{\Gamma' \Rightarrow \Delta, \gamma_1, \alpha} \qquad \dfrac{\mathcal{D}_{12}}{\Gamma', \gamma_2 \Rightarrow \Delta, \alpha}}{\Gamma \Rightarrow \Delta, \alpha}\,(\textsc{li}) \qquad \dfrac{\mathcal{D}_2}{\alpha, \Gamma \Rightarrow \Delta}\,(\textsc{r}2)}{\Gamma \Rightarrow \Delta}\,(\textsc{c}). \tag{4}$$

for some multiset of formulas Γ' and propositional formulas γ_1 and γ_2 such that $\Gamma \equiv \Gamma' \cup \{\gamma_1 \to \gamma_2\}$. Derivation (4) can be transformed into the following derivation:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_{11}[\gamma_1 \to \gamma_2 \Rightarrow]}{\Gamma \Rightarrow \Delta, \gamma_1, \alpha} \qquad \dfrac{\mathcal{D}_2[\Rightarrow \gamma_1]}{\alpha, \Gamma \Rightarrow \Delta, \gamma_1}}{\Gamma \Rightarrow \Delta, \gamma_1} \qquad \dfrac{\dfrac{\mathcal{D}_{12}[\gamma_1 \to \gamma_2 \Rightarrow]}{\Gamma, \gamma_2 \Rightarrow \Delta, \alpha} \qquad \dfrac{\mathcal{D}_2[\gamma_2 \Rightarrow]}{\alpha, \Gamma, \gamma_2 \Rightarrow \Delta}}{\Gamma, \gamma_2 \Rightarrow \Delta}}{\Gamma, \gamma_1 \to \gamma_2 \Rightarrow \Delta}. \tag{5}$$

Although this derivation has two instances of the cut rule, by Lemma 27 both of them are applied to trees of lower depth than the trees to which the cut rule is applied in derivation (4). Hence, by the induction hypothesis, $A \vdash \Gamma, \gamma_1 \to \gamma_2 \Rightarrow \Delta$. By Lemma 32, $A \vdash \Gamma \Rightarrow \Delta$.

Next, assume that (R1)=(XU) and α is not a principal formula of this rule. Then the derivation (3) can be replaced by the cut-free derivation $\dfrac{\mathcal{D}_1}{\Gamma \Rightarrow \Delta}$ (XP). Similar argument can be made in the case when (R1)=(XP).

Case C. We will consider the following three subcases: $\alpha \equiv \alpha_1 \to \alpha_2$, $\alpha \equiv \alpha_1 \otimes \alpha_2$, and $\alpha \equiv \alpha_1 \oplus \alpha_2$ for some formulas α_1 and α_2.

If $\alpha \equiv \alpha_1 \to \alpha_2$ is principal formula in both (R1) and (R2) then (R1)=(RI) and (R2)=(LI). Hence, the derivation (3) has the form:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_{11}}{\Gamma, \alpha_1 \Rightarrow \Delta, \alpha_2}}{\Gamma \Rightarrow \Delta, \alpha_1 \to \alpha_2}\,(\text{RI}) \qquad \dfrac{\dfrac{\mathcal{D}_{21}}{\Gamma \Rightarrow \Delta, \alpha_1} \qquad \dfrac{\mathcal{D}_{22}}{\Gamma, \alpha_2 \Rightarrow \Delta}}{\alpha_1 \to \alpha_2, \Gamma \Rightarrow \Delta}\,(\text{LI})}{\Gamma \Rightarrow \Delta}\,(\text{C}) \tag{6}$$

that can be re-arranged into a derivation in which both cut rule applications have smaller rank:

$$\dfrac{\dfrac{\mathcal{D}_{21}}{\Gamma \Rightarrow \Delta, \alpha_1} \qquad \dfrac{\dfrac{\mathcal{D}_{11}}{\Gamma, \alpha_1 \Rightarrow \Delta, \alpha_2} \qquad \dfrac{\mathcal{D}_{22}[\alpha_1 \Rightarrow]}{\Gamma, \alpha_1, \alpha_2 \Rightarrow \Delta}}{\Gamma, \alpha_1 \Rightarrow \Delta}\,(\text{C})}{\Gamma \Rightarrow \Delta}\,(\text{C}).$$

Assume that $\alpha \equiv \alpha_1 \oplus \alpha_2$. Since α is the principal formula in (R1) and (R2), each of these rules is (XU). Thus, the left subtree of the derivation (3) has the form

$$\dfrac{\dfrac{\mathcal{D}_{11}}{\phi_{I_1} \Rightarrow \psi_{J_1}, \alpha_1} \qquad \dfrac{\mathcal{D}_{12}}{\chi_{I_1} \Rightarrow \eta_{J_1}, \alpha_2}}{\Gamma_1, \{\phi_i \oplus \chi_i\}_{i \in I_1} \Rightarrow \Delta_1, \{\psi_j \oplus \eta_j\}_{j \in J_1}, \alpha_1 \oplus \alpha_2}\,(\text{XU}), \tag{7}$$

and the right subtree of the same derivation has the form

$$\dfrac{\dfrac{\mathcal{D}_{21}}{\alpha_1, \phi_{I_2} \Rightarrow \psi_{J_2}} \qquad \dfrac{\mathcal{D}_{22}}{\alpha_2, \chi_{I_2}, \Rightarrow \eta_{J_2}}}{\Gamma_2, \alpha_1 \oplus \alpha_2, \{\phi_i \oplus \chi_i\}_{i \in I_2}, \Rightarrow \Delta_2, \{\psi_j \oplus \eta_j\}_{j \in J_2}}\,(\text{XU}), \tag{8}$$

where $\Gamma \equiv \Gamma_1, \{\phi_i \oplus \chi_i\}_{i \in I_1} \equiv \Gamma_2, \{\phi_i \oplus \chi_i\}_{i \in I_2}$ and $\Delta \equiv \Delta_1, \{\psi_j \oplus \eta_j\}_{j \in J_1} \equiv \Delta_2, \{\psi_j \oplus \eta_j\}_{j \in J_2}$. The derivations (7) and (8) could be re-arranged into the following two derivations:

$$\dfrac{\dfrac{\mathcal{D}_{11}[\phi_{I_2} \Rightarrow \psi_{J_2}]}{\phi_{I_1}, \phi_{I_2} \Rightarrow \psi_{J_1}, \psi_{J_2}, \alpha_1} \qquad \dfrac{\mathcal{D}_{21}[\phi_{I_1} \Rightarrow \psi_{J_1}]}{\alpha_1, \phi_{I_1}, \phi_{I_2} \Rightarrow \psi_{J_1}, \psi_{J_2}}}{\phi_{I_1}, \phi_{I_2} \Rightarrow \psi_{J_1}, \psi_{J_2}}\,(\text{C}),$$

$$\dfrac{\dfrac{\mathcal{D}_{12}[\chi_{I_2} \Rightarrow \eta_{J_2}]}{\chi_{I_1}, \chi_{I_2} \Rightarrow \eta_{J_1}, \eta_{J_2}, \alpha_2} \qquad \dfrac{\mathcal{D}_{22}[\chi_{I_1} \Rightarrow \eta_{J_1}]}{\alpha_2, \chi_{I_1}, \chi_{I_2} \Rightarrow \eta_{J_1}, \eta_{J_2}}}{\chi_{I_1}, \chi_{I_2} \Rightarrow \eta_{J_1}, \eta_{J_2}} \text{ (C)}.$$

Since those two derivations have cutformulas of smaller size than the original derivation, we can use induction hypothesis to conclude that $A \vdash \phi_{I_1}, \phi_{I_2} \Rightarrow \psi_{J_1}, \psi_{J_2}$ and $A \vdash \chi_{I_1}, \chi_{I_2} \Rightarrow \eta_{J_1}, \eta_{J_2}$. Hence, by (XU) rule, $A \vdash \Gamma_1, \Gamma_2, \{\phi_i \oplus \chi_i\}_{i\in I_1}, \{\phi_i \oplus \chi_i\}_{i\in I_2} \Rightarrow \Delta_1, \Delta_2, \{\psi_j \oplus \eta_j\}_{j\in J_1}, \{\psi_j \oplus \eta_j\}_{j\in J_2}$. In other words, $\Gamma, \Gamma \vdash \Delta, \Delta$. Therefore, by Lemma 32, $A \vdash \Gamma \Rightarrow \Delta$.

Assume $\alpha \equiv \alpha_1 \otimes \alpha_2$. Since α is the principal formula in rules (R1) and (R2), each of them is (XP). Also, the premises of the rule (R1) could be combined into the following two groups:

$$\forall J_1' \subseteq J_1(A \vdash \phi_{I_1} \Rightarrow \psi_{J_1'} \text{ or } A \vdash \chi_{I_1} \Rightarrow \eta_{J_1 \setminus J_1'}, \alpha_2) \tag{9}$$

$$\forall J_1' \subseteq J_1(A \vdash \phi_{I_1} \Rightarrow \psi_{J_1'}, \alpha_1 \text{ or } A \vdash \chi_{I_1} \Rightarrow \eta_{J_1 \setminus J_1'}) \tag{10}$$

and premises of rule (R2) are:

$$\forall J_2' \subseteq J_2(A \vdash \alpha_1, \phi_{I_2} \Rightarrow \psi_{J_2'} \text{ or } A \vdash \alpha_2, \chi_{I_2} \Rightarrow \eta_{J_2 \setminus J_2'}) \tag{11}$$

where $\Gamma \equiv \Gamma_1, \{\phi_i \otimes \chi_i\}_{i\in I_1} \equiv \Gamma_2, \{\phi_i \otimes \chi_i\}_{i\in I_2}$ and $\Delta \equiv \Delta_1, \{\psi_j \otimes \eta_j\}_{j\in J_1} \equiv \Delta_2, \{\psi_j \otimes \eta_j\}_{j\in J_2}$. We need to show that $A \vdash \Gamma \Rightarrow \Delta$. By Lemma 32, it is sufficient to show that $A \vdash \Gamma, \Gamma \Rightarrow \Delta, \Delta$. Furthermore, according to the (XP) rule, this would follow from provability in A of the following group of sequents:

$$\forall J_1' \subseteq J_1, \forall J_2' \subseteq J_2(\phi_{I_1}, \phi_{I_2} \Rightarrow \psi_{J_1'}, \psi_{J_2'} \text{ or } \chi_{I_1}, \chi_{I_2} \Rightarrow \eta_{J_1 \setminus J_1'}, \eta_{J_2 \setminus J_2'}).$$

We will show, by contradiction, that the above sequents are provable in A. Indeed, let for some $J_1' \subseteq J_1$ and $J_2' \subseteq J_2$:

$$\nvdash_A \phi_{I_1}, \phi_{I_2} \Rightarrow \psi_{J_1'}, \psi_{J_2'}, \qquad \nvdash_A \chi_{I_1}, \chi_{I_2} \Rightarrow \eta_{J_1 \setminus J_1'}, \eta_{J_2 \setminus J_2'}. \tag{12}$$

Hence, by Lemma 27, $\nvdash_A \phi_{I_1} \Rightarrow \psi_{J_1'}$ and $\nvdash_A \chi_{I_1} \Rightarrow \eta_{J_1 \setminus J_1'}$. Combining the last statement with (9) and (10) we can conclude that $A \vdash \chi_{I_1} \Rightarrow \eta_{J_1 \setminus J_1'}, \alpha_2$ and $A \vdash \phi_{I_1} \Rightarrow \psi_{J_1'}, \alpha_1$. Thus, by Lemma 27,

$$A \vdash \chi_{I_1}, \chi_{I_2} \Rightarrow \eta_{J_1 \setminus J_1'}, \eta_{J_2 \setminus J_2'}, \alpha_2, \qquad A \vdash \phi_{I_1}, \phi_{I_2} \Rightarrow \psi_{J_1'}, \psi_{J_2'}, \alpha_1. \tag{13}$$

However, by the same Lemma 27, we can conclude from (11) that either $A \vdash \alpha_1, \phi_{I_1}, \phi_{I_2} \Rightarrow \psi_{J_1'}, \psi_{J_2'}$ or $A \vdash \alpha_2, \chi_{I_1}, \chi_{I_2} \Rightarrow \eta_{J_1 \setminus J_1'}, \eta_{J_2 \setminus J_2'}$. Hence, taking into account (13), we can say that either sequent $\phi_{I_1}, \phi_{I_2} \Rightarrow \psi_{J_1'}, \psi_{J_2'}$ or sequent $\chi_{I_1}, \chi_{I_2} \Rightarrow \eta_{J_1 \setminus J_1'}, \eta_{J_2 \setminus J_2'}$ is provable in A^+ with just a single use of the Cut rule. In the case of the first sequent, the cutformula is α_1 and in the case of the second sequent the cutformula is α_2, either of them has a smaller size than $\alpha_1 \otimes \alpha_2$. Thus, by the induction hypothesis, either $A \vdash \phi_{I_1}, \phi_{I_2} \Rightarrow \psi_{J_1'}, \psi_{J_2'}$ or $A \vdash \chi_{I_1}, \chi_{I_2} \Rightarrow \eta_{J_1 \setminus J_1'}, \eta_{J_2 \setminus J_2'}$. Contradiction to (12). $\square$

6 Hilbert Axiomatics

Definition 35 *Hilbert version* **HS** *of basic Subtyping Logic is the minimal, closed under Modus Ponens, extension of Classical Propositional Logic by the following axioms and inference rules for Cartesian product and disjoint union:*

(1) $\bot \otimes \phi \to \bot, \qquad \phi \otimes \bot \to \bot, \qquad \bot \oplus \bot \to \bot,$
(2) $(\phi \to \psi) \otimes (\chi \to \tau) \to (\phi \otimes \chi \to \psi \otimes \tau),$
(3) $(\phi \to \psi) \oplus (\chi \to \tau) \to (\phi \oplus \chi \to \psi \oplus \tau),$
(4) $\mu \otimes \nu \to [(\phi \otimes \nu \to \psi \otimes \nu) \to (\phi \to \psi) \otimes \nu],$
(5) $\mu \otimes \nu \to [(\mu \otimes \phi \to \mu \otimes \psi) \to \mu \otimes (\phi \to \psi)],$
(6) $\mu \oplus \nu \to [(\phi \oplus \chi \to \psi \oplus \tau) \to (\phi \to \psi) \oplus (\chi \to \tau)],$
(7) $\phi \to \psi, \chi \to \tau \vdash \phi \otimes \chi \to \psi \otimes \tau,$
(8) $\phi \to \psi, \chi \to \tau \vdash \phi \oplus \chi \to \psi \oplus \tau.$

Theorem 5 *For any two multisets of modal propositional formulas* Γ *and* Δ, $\mathbf{S} \vdash \Gamma \Rightarrow \Delta$ *if and only if* $\mathbf{HS} \vdash \bigwedge \Gamma \to \bigvee \Delta$. $\square$

7 Conclusions

We have developed a sequential logical calculus $\mathbf{S}_\omega$ that describes subtyping properties of mutually-recursive types built using Cartesian product and disjoint union. From logical prospective, this calculus is an extension of the classical propositional logic by just two new binary modalities: $\otimes$ and $\oplus$. We have proved its completeness with respect to all algebras of terms. In case when the set of fixed point equations is empty, this calculus becomes basic logic of subtyping **S**. We proved Cut elimination theorem and gave Hilbert-style axiomatic for **S**. We also investigated the case of well-founded universes. It turns out that in this case the basic logic of subtyping (without recursive types) stays the same, but subtyping logic of recursive types needs an extra well-foundness inference rule for completeness. Subtyping logic of mutually-recursive types over well-founded universe $\mathbf{S}_\rho^\omega$ is proven to be decidable. The same question about logic $\mathbf{S}_\rho$ of all mutually-recursive types remains open.

The most logical next step in this work is the description of subtyping logics of inductive and co-inductive types. Those are logics describing subtyping properties of the least and the greatest solutions of the appropriate system of type equations. Another direction of the future research is the extension of subtyping logic **S** by modalities corresponding to the other type constructors: constructive and non-constructive function, quotient, dependent product, etc.

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