On Modal Logic of Deductive Closure

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Abstract

A new modal logic $\mathcal{D}$ is introduced. It describes properties of provability by interpreting modality as a deductive closure operator on sets of formulas. Logic $\mathcal{D}$ is proven to be decidable and complete with respect to this semantics.

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1 Introduction

The language of modal logics can be used to capture general properties of provability. [Gödel, 1933] suggested that modal logic $S4$ specifies such properties, although he had never given a precise meaning to this statement. The well-developed approach to describing properties of provability in the modal language, known the *logic of provability*, is based on interpreting modal formulas as statements in a theory T and interpreting modality operator as a provability predicate in this theory. That is, statement $\Box\phi$ is interpreted as "ϕ is provable". [Solovay, 1976] described provability logic of Peano Arithmetic and has proven its decidability. A review of related results could be found in [Boolos, 1993]. [Artëmov, 1994] introduced a *logic of proofs* – a logical system with labeled modalities in which $\Box_p\phi$ is interpreted as "p is a proof of ϕ".

In order to define provability logic of a theory T, the language of T should be sufficiently rich to express the provability predicate. In addition, most results on the provability logic assume that theory T is powerful enough to prove

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different "basic" properties of such predicate. The most common assumption is that T includes, in some form, the language and the axioms of Peano Arithmetic. The notion of provability, of course, is not restricted to rich theories. This paper suggests an alternative approach to provability interpretation of modal logics that puts almost no restrictions on theory T.

Our approach is based on the set-theoretical interpretation of propositional logic that maps propositional connectives conjunction $\wedge$, disjunction $\vee$, and negation $\neg$ into set operations intersection $\cap$, union $\cup$, and complement $\complement_\Sigma$, where Σ is some universe. It is a well-known observation that a formula is provable in the classical propositional logic if and only if for any choice of universe Σ and for any interpretation of propositional variables as subsets of Σ, the value of the formula is the entire universe.

Several possible extensions of the classical logic by modal operators corresponding, under the above set semantics, to additional set operations have been considered before. McKinsey and Tarski [1944] established that if the universe U is a topological space, then modal logic S4 describes properties of the interior operator. If the universe U is the set of all words in some alphabet, then properties of the logical connectives corresponding to product and star operations are axiomatized by Interval Temporal Logic [Moszkowski and Manna, 1984]. In [Naumov, 2005a], the author describes an extension of the classical propositional logic by binary modalities, corresponding to the operations disjoint union and Cartesian product and in [Naumov, 2005b] the binary modality corresponding to the type of partial recursive functions.

In this paper, we investigate modal logic defined by the deductive closure operator. Namely, we assume that the universe Σ is a set of statements in the language of theory T. Then we can interpret the modality $\Box$ as the deductive closure operator in the universe Σ with respect to provability in theory T. The set of all modal formulas whose interpretation is equal to the entire universe Σ, no matter what the choice of the universe and the interpretation of propositional variables are, will be called the logic of deductive closure of theory T.

Defined in such a way the logic of deductive closure is not quite typical modal logic, because it does not contain modal distributivity axiom $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$. Thus, one would not expect it to have any reasonable Kripke semantics. Yet, we will be able to provide a finite, sound, and complete Kripke-style semantics for this logic.

The logic of deductive closure is not meant to replace the logic of provability, but rather to complement it. Those two logics use modal language to capture different kinds of properties of provability. While the logic of provability deals with reflection – the ability of theory T to reason about itself, the logic of deductive closure focuses on properties of deductive closure as a set operation.

Throughout the rest of the paper we assume that T is just the classical propositional logic. Logics of deductive closure for some other deductive systems are briefly discussed in the conclusion.

2 Language

Definition 1 *Language $\mathcal{L}$ of the logic of deductive closure consist of propositional variables, boolean connectives $\rightarrow$ and $\perp$, and modality $\Box$.*

As usual, $\neg\phi$ and $\phi \vee \psi$ are considered to be abbreviations for $\phi \rightarrow \perp$ and $(\neg\phi) \rightarrow \psi$ correspondingly. We use letters $p, q, \dots$ to denote propositional variables of $\mathcal{L}$ and letters $\phi, \psi, \chi, \dots$ to represent formulas of $\mathcal{L}$. In addition, we will consider modality-free language $\mathcal{L}_0$ of the classical propositional logic. Letters $a, b, \dots$ denote propositional variables of $\mathcal{L}_0$ and letters $\alpha, \beta, \gamma, \dots$ stand for formulas of $\mathcal{L}_0$.

Definition 2 *Let Σ be a set of statements in language $\mathcal{L}_0$. A Σ-valuation is an arbitrary map $*$ of propositional variables of language $\mathcal{L}$ into set Σ.*

We also will be dealing with boolean valuations of formulas in language $\mathcal{L}_0$. Symbol $\star$ will be used for such valuations.

Definition 3 *An arbitrary Σ-valuation $*$ could be extended on all formulas in $\mathcal{L}$ as follows:*

(1) $\perp^* = \varnothing$,
(2) $(\phi \rightarrow \psi)^* = \complement_\Sigma(\phi^*) \cup \psi^*$,
(3) $(\Box\phi)^* = \{\alpha \in \Sigma \mid \phi^* \vdash \alpha\}$, *where $\vdash$ denotes provability in the classical propositional logic.*

3 Axioms

Definition 4 *In addition to classical propositional tautologies and Modes Ponens inference rule, logic of deductive closure $\mathcal{D}$ contains the following axioms and inference rules:*

- *reflexivity:* $\phi \rightarrow \Box\phi$,
- *transitivity:* $\Box(\phi \vee \Box\phi) \rightarrow \Box\phi$,
- *monotonicity:* $\dfrac{\phi \rightarrow \psi}{\Box\phi \rightarrow \Box\psi}$.

We will write $\vdash_{\mathcal{D}} \phi$ to state that ϕ is provable in $\mathcal{D}$. At the same time, notation $\Gamma \vdash \alpha$ will refer to provability of α in the *classical propositional logic* from a list of hypotheses Γ.

Theorem 1 *If $\vdash_{\mathcal{D}} \phi$ then for any interpretation $*$ over an arbitrary set of statements Σ, $\phi^* = \Sigma$.*

Proof. Induction on the size of the derivation in $\mathcal{D}$. Soundness of classical propositional tautologies as well as that of Modes Ponens are trivial. We will show soundness of reflexivity and transitivity axioms and monotonicity inference rule:

(1) If $\alpha \in \phi^*$, then $\phi^* \vdash \alpha$. Thus, $\alpha \in (\Box\phi)^*$.
(2) Suppose that $\alpha \in (\Box(\phi \vee \Box\phi))^*$. Hence, α is provable from the union of ϕ^* and all theorems derivable from ϕ^*. Thus, α should be provable from ϕ^* only. Therefore, $\alpha \in (\Box\phi)^*$.
(3) Let $(\phi \rightarrow \psi)^* = \Sigma$. Hence, $\phi^* \subseteq \psi^*$. Thus, $\{\alpha \in \Sigma \mid \phi^* \vdash \alpha\} \subseteq \{\alpha \in \Sigma \mid \psi^* \vdash \alpha\}$. Then, $(\Box\phi)^* \subseteq (\Box\psi)^*$. Therefore, $(\Box\phi \rightarrow \Box\psi)^* = \Sigma$. $\Box$

4 Kripke-style Model

Usually, Kripke-style semantics is considered only for so called *normal*[1] modal logics. Although $\mathcal{D}$ is not normal, in this section we will be able to define some kind of Kripke-style semantics for $\mathcal{D}$ and prove its soundness completeness with respect to this semantics. This result will be used later to show completeness of $\mathcal{D}$ with respect to deductive closure semantics defined above.

Definition 5 *Kripke model is a triple $\langle W, \lhd, \Vdash \rangle$, where W is a set of "possible worlds", $\lhd$ is an "accessibility" relation between elements of W and subsets of W, and $\Vdash$ is a "forcing" relation between elements of W and propositional variables of $\mathcal{L}$. Relation $\lhd$ is assumed to satisfy the following two properties:*

(1) reflexivity: if $x \in Y$, then $x \lhd Y$,
(2) transitivity: if $x \lhd Y$ and $y \lhd Z$ for all $y \in Y$, then $x \lhd Z$.

Definition 6 *Relation $\Vdash$ can be extended to the relation between worlds and arbitrary $\mathcal{L}$ formulas as follows:*

(1) $w \nVdash \bot$,
(2) $w \Vdash \phi \rightarrow \psi$ if and only if $w \nVdash \phi$ or $w \Vdash \psi$,
(3) $w \Vdash \Box\phi$ if and only if $\exists V(w \lhd V \wedge \forall v \in V(v \Vdash \phi))$.

[1] For definition, see, for example Hughes and Cresswell [1996].

Note that, in the above definition, if $w \lhd \varnothing$, then $w \Vdash \Box\phi$ for any modal formula ϕ.

Lemma 1 *For any world w of a Kripke model $\langle W, \lhd, \Vdash\rangle$ and any two subsets X and Y of W, if $w \lhd X$ and $X \subseteq Y$, then $w \lhd Y$.*

Proof. By the reflexivity of relation $\lhd$, we have $x \lhd Y$ for any $x \in X$. Thus, the transitivity of relation $\lhd$ implies that $w \lhd Y$. $\Box$

Theorem 2 *Let ϕ be an arbitrary modal formula. If $\vdash_{\mathcal{D}} \phi$, then $w \Vdash \phi$ for any world $w \in W$ of a Kripke model $\langle W, \lhd, \Vdash\rangle$.*

Proof. Induction on the length of derivation in $\mathcal{D}$. Cases of classical propositional logic tautologies and Modus Ponens inference rule are trivial. We will only consider modal axioms and inference rules of logic $\mathcal{D}$.

To show that $w \Vdash \phi \to \Box\phi$, assume that $w \Vdash \phi$. Consider $V = \{w\}$. By reflexivity of relation $\lhd$, we have $w \lhd V$. At the same time, $\forall v \in V(v \Vdash \phi)$. Thus, $w \Vdash \Box\phi$.

To show that $w \Vdash \Box(\phi \vee \Box\phi) \to \Box\phi$, suppose that $w \Vdash \Box(\phi \vee \Box\phi)$. Thus, there is subset $V \subseteq W$ such that $w \lhd V$ and $\forall v \in V(v \Vdash \phi \vee \Box\phi)$. In other words, $\forall v \in V(v \Vdash \phi$ or $v \Vdash \Box\phi)$. It has been show in the previous paragraph that formula $\phi \to \Box\phi$ is forced in every world of any Kripke model. Hence, $\forall v \in V(v \Vdash \Box\phi)$. By Definition 6,

$$\forall v \in V \exists X_v (v \lhd X_v \wedge \forall x \in X_v (x \Vdash \phi)). \tag{1}$$

Note that $\forall x \in X_v(x \Vdash \phi)$ implies that $X_v \subseteq \{y \in W \mid y \Vdash \phi\}$. Thus, by Lemma 1, Statement (1) implies that $\forall v \in V(v \lhd \{y \in W \mid y \Vdash \phi\})$. Hence, by transitivity of relation $\lhd$, we have $w \lhd \{y \in W \mid y \Vdash \phi\}$. Therefore, by Definition 6, $w \Vdash \Box\phi$.

Finally, we will show that if formula $\phi \to \psi$ is forced in every world of the Kripke model, then $w \Vdash \Box\phi \to \Box\psi$. Suppose that $w \Vdash \Box\phi$. Then there is a subset V of W such that $w \lhd V$ and $\forall v \in V(v \Vdash \phi)$. Since $\phi \to \psi$ is forced in every world, $\forall v \in V(v \Vdash \psi)$. Therefore, $w \Vdash \Box\psi$. $\Box$

Definition 7 *Kripke model $\langle W, \lhd, \Vdash\rangle$ is finite if set W is finite.*

Theorem 3 *If $\nvdash_{\mathcal{D}} \phi$ then there is a finite Kripke model $\langle W, \lhd, \Vdash\rangle$ and a world w_0 of this model such that $w_0 \nVdash \phi$.*

Proof. Assume that $\nvdash_{\mathcal{D}} \phi$. Let S be the set of all subformulas of ϕ and

$$\bar{S} = S \cup \{\neg\phi \mid \phi \in S\}.$$

Let W be the set of all maximal $\mathcal{D}$-consistent subsets of $\bar{S}$. We define $w \lhd V$ if for any ψ such that $\neg\Box\psi \in w$ there is a $v \in V$ such that $\neg\psi, \neg\Box\psi \in v$. Let also $w \Vdash p$ be true if and only if $p \in w$.

Lemma 2 *Triple $\langle W, \lhd, \Vdash\rangle$ is a finite Kripke model.*

Proof. Set $W \subseteq 2^{\bar{S}}$. is finite. To show that relation $\lhd$ is reflexive, assume that $x \in Y$ and $\neg\Box\psi \in x$, then, by maximality of set x and reflexivity axiom, $\neg\phi \in x$. Thus, $x \lhd Y$.

To show that relation $\lhd$ is transitive, suppose that $x \lhd Y$ and $y \lhd Z$ for all $y \in Y$. We will show that $x \lhd Z$. Indeed, if $\neg\Box\psi \in x$, then there is an element $y_0 \in Y$ such that $\neg\Box\psi \in y_0$. Thus, there is an element $z_0 \in Z$ such that $\neg\psi, \neg\Box\psi \in z_0$. Therefore, $x \lhd Z$. $\Box$

Lemma 3 *For any $\Box\psi \in S$ and $x \in W$, if $\Box\psi \in x$, then there is a set of worlds Y such that $x \lhd Y$ and $\psi \in y$ for any $y \in Y$.*

Proof. First, we will show that for any $(\neg\Box\chi) \in x$, set $\{\neg\chi, \neg\Box\chi, \psi\}$ is consistent. By contradiction. If $\vdash_{\mathcal{D}} \psi \to \chi \vee \Box\chi$ then by monotonicity rule, $\vdash_{\mathcal{D}} \Box\psi \to \Box(\chi \vee \Box\chi)$. Taking into account transitivity axiom, we get $\vdash_{\mathcal{D}} \Box\psi \to \Box\chi$. Therefore, x is not consistent.

For any $(\neg\Box\chi) \in x$ consider set y_χ which is a maximal $\mathcal{D}$-consistent extension of set $\{\neg\chi, \neg\Box\chi, \psi\}$. We are only left to notice that $x \lhd \{y_\chi \mid (\neg\Box\chi) \in x\}$. $\Box$

Lemma 4 *For any $\psi \in S$ and any $w \in W$,*

$$\psi \in w \quad \Longleftrightarrow \quad w \Vdash \psi$$

Proof. Induction on the structural complexity of formula ψ. Cases when ψ is $\perp$ or a propositional variable follow from Definition 6 and the definition of $\Vdash$ on propositional variables.

(1) Let $\psi \equiv \psi_1 \to \psi_2$. $(\Rightarrow)$: If $w \nVdash \psi_1 \to \psi_2$ then $w \Vdash \psi_1$ and $w \nVdash \psi_2$. By the induction hypothesis, $\psi_1 \in w$ and $\psi_2 \notin w$. Since w is consistent, $\psi_1 \to \psi_2 \notin w$. $(\Leftarrow)$: Assume $w \Vdash \psi_1 \to \psi_2$. Thus, $(w \nVdash \psi_1) \vee (w \Vdash \psi_2)$. By the induction hypothesis, $(\psi_1 \notin w) \vee (\psi_2 \in w)$. By the maximality of w, $\phi_1 \to \phi_2 \in w$.

(2) Suppose $\psi \equiv \Box\chi$. $(\Rightarrow)$: If $\Box\chi \in w$ then, by Lemma 3, there is $Y \subseteq W$ such that $w \lhd Y$ and $\chi \in y$ for every $y \in Y$. By the induction hypothesis, $y \Vdash \chi$ for all $y \in Y$. Thus, by Definition 6, $w \Vdash \Box\chi$. $(\Leftarrow)$: Assume that $\Box\chi \notin w$. Since w is maximal, $\neg\Box\chi \in w$. Thus, by the definition of $\lhd$, for any Y such that $w \lhd Y$ there is $y \in Y$ such that $\neg\chi \in y$. By the induction hypothesis, it means that for any Y such that $w \lhd Y$ there is $y \in Y$ such that $y \nVdash \chi$. This, by Definition 6, implies that $w \nVdash \Box\chi$. $\Box$

We are ready to finish the proof of Theorem 3. Let w_0 be a maximal $\mathcal{D}$-consistent extension of the set $\{\neg\phi\}$. By Lemma 4, $w_0 \nVdash \phi$. $\square$

Corollary 1 *Logic $\mathcal{D}$ is decidable.*

5 Fixed Point Theorem

In this section we establish a technical result about classical propositional logic, which later will be used to convert a Kripke model of $\mathcal{D}$ into a Σ-valuation.

Definition 8 *For any two boolean functions $f, g : \{0,1\}^n \mapsto \{0,1\}$, let $f \leq g$ mean that $f(x_1, \ldots, x_n) \leq g(x_1, \ldots, x_n)$ for any set of boolean arguments $(x_1, \ldots, x_n)$.*

Definition 9 *Boolean function f is monotonic if $f(x_1, \ldots, x_n) \leq f(y_1, \ldots, y_n)$ for any two sets of boolean arguments $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ such that $x_i \leq y_i$ for any $i = 1, \ldots, n$.*

Definition 10 *Boolean function $f(x_1, \ldots, x_n)$ is x_i-sufficient if*

$$f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) \equiv 1.$$

Every propositional formula α in language $\mathcal{L}_0$ represents a boolean function. We denote this boolean function $\bar{\alpha}$. We will use names of propositional variables in α as names of arguments of function $\bar{\alpha}$.

Theorem 4 *Let $\beta_1(a_1, \ldots, a_n), \ldots, \beta_n(a_1, \ldots, a_n)$ be $\mathcal{L}_0$ formulas. If for any $i \in \{1, \ldots, n\}$ function $\bar{\beta}_i$ is monotonic and a_i-sufficient then there are propositional formulas $\gamma_1, \ldots, \gamma_n$ such that*

(1) $\vdash \gamma_i \leftrightarrow \beta_i[\gamma_1/a_1, \ldots, \gamma_n/a_n]$, for any $i = 1, \ldots, n$,
(2) If $\star$ is such boolean valuation that $\forall i(\beta_i^\star = a_i^\star)$, then $\forall i(\gamma_i^\star = a_i^\star)$.

Proof. Let us define formulas γ_i^k for $k = 0, \ldots$ and $i = 1, \ldots, n$ as follows

$$\gamma_i^k = \begin{cases} a_i & \text{if } k = 0 \\ \beta_i[\gamma_1^{k-1}/a_1, \ldots, \gamma_n^{k-1}/a_n] & \text{if } k > 0 \end{cases}$$

Lemma 5 *$\bar{\gamma}_i^k \leq \bar{\gamma}_i^{k+1}$ for $1 \leq i \leq n$ and $0 \leq k$.*

Proof. Induction on k.

(1) If $\bar{\gamma}_i^0(a_1,\ldots,a_n)=1$ then, by the definition of γ_i^0, $a_i=1$. Thus, by the definition of γ_i^k and taking into account that $\bar{\beta}_i$ is an a_i-sufficient boolean function, we have

$$\bar{\gamma}_i^1(a_1,\ldots,a_n)=\bar{\beta}_i(a_1,\ldots,a_{i-1},1,a_{i+1},\ldots,a_n)=1.$$

(2) If $\bar{\gamma}_i^k \leq \bar{\gamma}_i^{k+1}$ for any i then, by monotonicity of $\bar{\beta}_i$,

$$\bar{\beta}_i(\bar{\gamma}_1^k,\ldots,\bar{\gamma}_n^k) \leq \bar{\beta}_i(\bar{\gamma}_1^{k+1},\ldots,\bar{\gamma}_n^{k+1}).$$

Therefore, $\bar{\gamma}_i^{k+1} \leq \bar{\gamma}_i^{k+2}$. □

Since the domain of boolean functions $\bar{\gamma}_i^k$ is finite, it follows from the above lemma that all but first finitely many elements of infinite chain $\bar{\gamma}_i^0 \leq \bar{\gamma}_i^1 \leq \ldots$ are equal as boolean functions. Let us consider such k_0 that $\bar{\gamma}_i^{k_0}=\bar{\gamma}_i^{k_0+1}$ for any $i=1,\ldots,n$. Since any two propositional formulas representing the same boolean function are provably equivalent in the classical propositional logic, $\vdash \gamma_i^{k_0} \leftrightarrow \gamma_i^{k_0+1}$ for any $i=1,\ldots,n$. Let $\gamma_i=\gamma_i^{k_0}$ then, by the definition of γ_i^k,

$$\vdash \gamma_i \leftrightarrow \beta_i[\gamma_1/a_1,\ldots,\gamma_n/a_n] \text{ for any } i=1,\ldots,n.$$

Therefore, the first claim of Theorem 4 is established. Let now $\star$ be some boolean valuation of propositional variables such that $\beta_i^\star=a_i^\star$.

Lemma 6 *For any $i=1,\ldots,n$ and any $k=0,1,\ldots,$*

$$(\gamma_i^k)^\star=a_i^\star.$$

Proof. Induction on k. If $k=0$ then γ_i^k is a_i and the statement of the lemma is trivially true. If $k>0$ then, taking into account the induction hypothesis,

$$(\gamma_i^k)^\star=(\beta_i(\gamma_1^{k-1},\ldots,\gamma_n^{k-1}))^\star=\bar{\beta}_i((\gamma_1^{k-1})^\star,\ldots,(\gamma_n^{k-1})^\star)=\bar{\beta}_i(a_1^\star,\ldots,a_n^\star)=\beta_i^\star.$$

Finally, since $\beta_i^\star=a_i^\star$, we have $(\gamma_i^k)^\star=a_i^\star$. □

It follows from the above lemma that $(\gamma_i^{k_0})^\star=a_i^\star$ for any $i=1,\ldots,n$. Thus, $(\gamma_i)^\star=a_i^\star$. This ends the proof of Theorem 4. □

6 Completeness

This section concludes the proof of logic $\mathcal{D}$'s completeness with respect to the deductive closure semantics. It is done by converting a Kripke model into a Σ-valuation, where set of propositions Σ is constructed using Fixed Point Theorem of the previous section.

Theorem 5 *If $\nvdash_{\mathcal{D}} \phi$, then there is a finite set of propositional formulas Σ and a Σ-valuation $*$ such that $\phi^* \neq \Sigma$.*

Proof. Assume that $\nvdash_{\mathcal{D}} \phi$. By Theorem 3, there is a finite Kripke model $\langle W, \lhd, \Vdash \rangle$ such that $w_0 \nVdash \phi$ for some $w_0 \in W$. We will identify a unique propositional variable a_w with every world of this model. Let us consider propositional formulas $\{\beta_w \mid w \in W\}$ such that

$$\beta_w = \bigvee_{w \lhd V} \bigwedge_{v \in V} a_v.$$

Lemma 7 *For any $w \in W$, boolean function $\bar{\beta}_w$ is monotonic.* $\square$

Lemma 8 *For any $w \in W$, boolean function $\bar{\beta}_w$ is a_w-sufficient.*

Proof. If $(a_w)^\star$ is true, then so is $(\bigwedge_{v \in \{w\}} a_w)^\star$. By the reflexivity of $\lhd$,

$$(\bigvee_{w \lhd V} \bigwedge_{v \in V} a_v)^\star$$

is also true. $\square$

By Theorem 4, there are propositional formulas $\{\gamma_w \mid w \in W\}$ such that

$$\vdash \gamma_w \leftrightarrow \bigvee_{w \lhd V} \bigwedge_{v \in V} \gamma_v \tag{2}$$

$$\forall \star\, (\forall w (a_w^\star = (\bigvee_{w \lhd V} \bigwedge_{v \in V} a_v)^\star) \to \forall w (\gamma_w^\star = a_w^\star)) \tag{3}$$

Lemma 9 *$x \lhd Y$ if and only if $\{\gamma_y \mid y \in Y\} \vdash \gamma_x$.*

Proof. $(\Rightarrow)$: If $x \lhd Y$ then, according to equivalence (2), $\vdash \bigwedge_{y \in Y} \gamma_y \to \gamma_x$. Thus, $\{\gamma_y \mid y \in Y\} \vdash \gamma_x$. $(\Leftarrow)$: Assume $\neg(x \lhd Y)$. Consider boolean valuation $\star$ of propositional variables such that a_z is true if and only if $z \lhd Y$. To show that $\{\gamma_y \mid y \in Y\} \nvdash \gamma_x$, it will be sufficient to show that $\gamma_y^\star$ is true for all $y \in Y$ and that $\gamma_x^\star$ is false.

Proposition 1 *For any world $w \in W$,*

$$a_w^\star = (\bigvee_{w \lhd V} \bigwedge_{v \in V} a_v)^\star.$$

Proof. First, if $a_w^\star$ is true then, by the definition of $\star$, $w \lhd Y$. By the reflexivity of $\lhd$, $y \lhd Y$ for all $y \in Y$. Thus, $(\bigwedge_{y \in Y} a_y)^\star$ is true. Since $w \lhd Y$, $(\bigvee_{w \lhd V} \bigwedge_{v \in V} a_v)^\star$ is also true. Second, if $(\bigvee_{w \lhd V} \bigwedge_{v \in V} a_v)^\star$ is true then there is some V_0 such that $w \lhd V_0$ and $(\bigwedge_{v \in V_0} a_v)^\star$ is true. In other worlds, $a_v^\star$ is true for all $v \in V_0$. By the definition of $\star$ it means that $v \lhd Y$ for all $v \in V_0$. Thus, by the transitivity of $\lhd$, $w \lhd Y$. Therefore, by the definition of $\star$, $a_w^\star$ is true. $\square$

Let us go back to the proof of Lemma 9. Taking into account Proposition 1 and implication (3), it would be sufficient to show that $a_y^\star$ is true for all $y \in Y$ and $a_x^\star$ is false. The first statement is true by the definition of $\star$ and the reflexivity of $\lhd$. The second statement follows from the assumption $\neg(x \lhd Y)$ and the definition of $\star$. $\Box$

Definition 11 *Let* $\Sigma = \{\gamma_w \mid w \in W\}$ *and* $*$ *be such* Σ*-valuation that* $p^* = \{\gamma_w \mid w \Vdash p\}$.

Lemma 10 $w \Vdash \psi$ *if and only if* $\gamma_w \in \psi^*$.

Proof. Induction on the structural complexity of ψ. The case when ψ is a propositional symbol immediately follows from Definition 11. The case when ψ is symbol $\bot$ follows from Definition 6 and Definition 3.

(1) Assume that ψ is $\psi_1 \to \psi_2$. By Definition 6, $w \Vdash \psi$ is equivalent to $(w \nVdash \psi_1) \vee (w \Vdash \psi_2)$, which, by the induction hypothesis, is equivalent to $\gamma_w \in \complement(\psi_1^*) \cup \psi_2^*$. The last, by Definition 3, is equivalent to $\gamma_w \in (\psi_1 \to \psi_2)^*$.

(2) Suppose that ψ is $\Box\chi$. $(\Rightarrow)$: If $w \Vdash \Box\chi$ then there is V such that $w \lhd V$ and $v \Vdash \chi$ for any $v \in V$. Thus, by Lemma 9, $\{\gamma_v \mid v \in V\} \vdash \gamma_w$ and, by the induction hypothesis, $\gamma_v \in \chi^*$ for any $v \in V$. Hence, $\chi^* \vdash \gamma$. Therefore, by Definition 3, $\gamma_w \in (\Box\chi)^*$. $(\Leftarrow)$: If $\gamma_w \in (\Box\chi)^*$ then, by Definition 3, $\chi^* \vdash \gamma_w$. Thus, by Lemma 9, $w \lhd \{v \mid \gamma_v \in \chi^*\}$. By the induction hypothesis, $w \lhd \{v \mid v \Vdash \chi\}$. Hence, by Definition 6, $w \Vdash \Box\chi$. $\Box$

The statement of Theorem 5 follows from the above lemma and the assumption $w_0 \nVdash \phi$. $\Box$

7 Conclusions

We have introduced a modal logic $\mathcal{D}$ describing deductive closure properties of the classical propositional logic, gave its complete axiomatization, and proved its decidability. It is clear that the same proof can be carried out for classical predicate calculus. A description of modal logic of deductive closure for other logical systems remains an open question. One would think that the same logic $\mathcal{D}$ describes deductive closure properties of an intuitionistic logic, but the proof, given in this paper, cannot be easily extrapolated to cover this case. Even more interesting are the cases of linear and non-monotonic logics whose modal logics of deductive closure are clearly different from $\mathcal{D}$.

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