A Proof-Theoretic Approach to the HOL-Nuprl Connection with Applications to Proof Translation*

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Abstract. The connection between the HOL and Nuprl proof assistants earlier introduced by Howe is reexamined from a proof-theoretic point of view as a translation between the sentences of HOL and a classical variant of Nuprl followed by a theory interpretation. Using standard terminology from the theory of general logics we establish a proof-theoretic correctness result of the translation which complements Howe's semantics-based argument. The correctness theorem is obtained in a constructive way by giving a Nuprl proof for each proof of a translated HOL theorem. As an application of this approach we have developed a proof translator that converts HOL proofs into their Nuprl counterparts.

1 Introduction

We investigate Doug Howe's connection [2, 6] between the classical logic of the HOL system [3] and a classical variant of the Nuprl type theory [1] from the viewpoint of general logics [8]. The main idea behind this connection is to rediscover the classical logic present in the boolean data type of an expressive type theory like Nuprl using a hybrid computational/set-theoretic semantics [5, 4]. This semantics justifies a classical variant of Nuprl (see [4] for details) which makes use of the axiom of the excluded middle. In the remainder of this article we will simply refer to this variant as Nuprl. The benefit of this approach is that an extensional and an intensional logic can coexist in a single framework and that we can take advantange of both. The former is well suited to deal directly with external classical knowledge, and for the latter theorem-proving is well supported in the Nuprl system and, moreover, it allows extraction of programs from proofs.

The HOL-Nuprl connection as it has been implemented in [7] can be understood in the framework of general logics as a composition of two stages:

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1. The first stage is a *translation* of an *axiomatic HOL theory* into an *axiomatic Nuprl theory*. The use of the term "axiomatic" emphasizes the fact that the theories are not necessarily only definitional extensions of the basic logic. The translation is, by its very nature, meta-logical, in the sense that, by relating two different logics, it is beyond the scope of each of them. It is therefore a critical stage whose correctness cannot be reduced to that of the two theorem provers involved and requires careful analysis.
2. The second stage is the *interpretation* of an axiomatic Nuprl theory inside Nuprl. In this way we can often obtain a *computationally meaningful* theory, which is closer to the spirit of Nuprl, that favors definitional extensions. As in Howe's extension of Nuprl, this interpretation stage can take place inside the Nuprl system in a formally rigorous way. However, since there are many possible choices for the interpretation, and certain proofs to be carried out, user interaction is required.

The present article tries to clarify the translation in terms of maps between entailment systems, which provide an abstract metalogical setting for formal systems within the theory of general logics [8]. The questions we address are: What are the entailment systems involved, and how can the translation be explained at an abstract level that makes its correctness easy to verify? The interpretation stage on the other hand does not take place between different logics but within a single logic, so that an actual interpretation can be formally verified in a rigorous way. The critial inter-logic translation stage should be kept as simple as possible, so that we can achive a high confidence that it is indeed correct. As a benefit of the proof-theoretic approach the correctness result in terms of entailment systems immediately suggests extending the HOL-Nuprl connection to a translator that does not only translate theorems but also their proofs. In the last section we report on some practical experience with an implementation of such a proof translator on the top of the Nuprl system.

2 Preliminaries

In the sequel we often make use of finite lists of objects. We use [] for the empty list and a comma to denote list concatenation. If σ is a meta-variable ranging over certain objects, σ^m is a meta-variable ranging over lists of length m of such objects. The elements of the list can be accessed by σ^m_i for $i \in \{1 \ldots m\}$. Sometimes objects have a certain structure, for instance, if they are of the form $x : \sigma$ we also write $x^n : \sigma^n$ for a list of length n of such objects. So we assume notational conventions such as $x^n : \sigma^m = x^n_1 : \sigma^n_1, \ldots, x^n_n : \sigma^n_n$.

3 HOL

HOL [3] is a proof development system based on higher-order logic. It uses an ML-style polymorphic lambda-calculus together with an axiomization of the logic using polymorphic equality, implication and Hilbert's epsilon operators as basic ingredients. As most higher-order logics it is a logic of total functions. The

HOL system favors conservative theory extensions (to introduce new constants and/or new data types) but axiomatic extensions are also supported.

The following presentation of HOL is close to [3] but we have cast it into a categorical setting which is suggested by the framework of general logics [8].

3.1 Syntactic Categories

We assume countably infinite disjoint sets of *constants*, *variables*, *type constants*, and *type variables*. Meta-variables for type constants and constants are ν and c, respectively. We use the meta-variables α, β, and γ for arbitrary but distinct type variables and meta-variables x, y, and z for arbitrary but distinct variables.

A *type constant declaration* takes the form $\nu : \mathsf{Type}^n \to \mathsf{Type}$ where ν is the *type constant* and n is its *arity*.[2] It is called an *atomic type declaration* if $n = 0$ (in this case we write $\nu : \mathsf{Type}$) and a *type operator declaration* otherwise. A *type signature* Ω is a finite set of type constant declarations with distinct type constants. We furthermore assume that a type signature Ω contains a *standard type constant declaration* $\triangleright\ \mathsf{o} : \mathsf{Type}$ and we refer to o as the *type of propositions*. The category of type signatures **HolTypeSign** has type signatures as objects, and its morphisms are functions mapping type constants in one signature to type constants in the other while preserving the standard type constants and the arity of each type constant.

A *type context* takes the form $\alpha^n : \mathsf{Type}^n$ such that all its type variables α_i^n are distinct. The category of type contexts is denoted by **HolTypeCtxt**. Its morphisms, called *type context inclusions*, are inclusions between the sets of declared type context variables. A *full type context* has the form $\Omega \triangleright \alpha^n : \mathsf{Type}^n$ and the category of full type contexts is $\mathbf{HolFullTypeCtxt} = \mathbf{HolTypeSign} \times \mathbf{HolTypeCtxt}$.

The set $\mathcal{T}^{Hol}$ of *types* over a type signature Ω is inductively defined as follows: **(1)** $\alpha^n : \mathsf{Type}^n \triangleright \alpha_i^n$ is in $\mathcal{T}^{Hol}$ for each $i \in \{1...n\}$. **(2)** If σ_i^m is in $\mathcal{T}^{Hol}$ for each $i \in \{1...m\}$ and $\nu : \mathsf{Type}^m \to \mathsf{Type}$ is a type constant declaration in Ω then $\alpha^n : \mathsf{Type}^n \triangleright \nu(\sigma^m)$ is in $\mathcal{T}^{Hol}$. We write ν instead of $\nu(\sigma^m)$ if $m = 0$. **(3)** If $\alpha^n : \mathsf{Type}^n \triangleright \sigma$ and $\alpha^n : \mathsf{Type}^n \triangleright \tau$ are in $\mathcal{T}^{Hol}$ then $\alpha^n : \mathsf{Type}^n \triangleright \sigma \to \tau$ is in $\mathcal{T}^{Hol}$. An expression σ is said to be a *type* over a full type context $\Omega \triangleright \alpha^n : \mathsf{Type}^n$ if $\alpha^n : \mathsf{Type}^n \triangleright \sigma$ is a type over Ω. The set of types over $\Omega \triangleright \alpha^n : \mathsf{Type}^n$ is denoted by $\mathit{HolType}(\Omega \triangleright \alpha^n : \mathsf{Type}^n)$. Each full type context morphism is lifted to a function between sets of types in the natural way. This defines a functor $\mathit{HolType} : \mathbf{HolFullTypeCtxt} \to \mathbf{Set}$. A *full type* has the form $\Omega \triangleright \alpha^n : \mathsf{Type}^n \triangleright \sigma$, where σ is a type over $\Omega \triangleright \alpha^n : \mathsf{Type}^n$.

Let Ω be a type signature. A *constant declaration* over Ω takes the form $\alpha^n : \mathsf{Type}^n \triangleright c : \sigma$, where σ is a type over $\Omega \triangleright \alpha^n : \mathsf{Type}^n$ and where each type variable α_i^n occurs in σ. Here c is a *constant* and σ is called the *type* of c. A *signature* Δ over Ω is a set of constant declarations over Ω with distinct constants. We assume that Δ contains *standard constant declarations* $\triangleright\ \Rightarrow : \mathsf{o} \to \mathsf{o} \to \mathsf{o}$

[2] Type is part of our suggestive notation, but it does not have a formal status in HOL.

and $\alpha : \mathsf{Type} \triangleright = : \alpha \to \alpha \to o$ for *logical implication* and *polymorphic equality*, respectively. A *signature morphism* between signatures Δ and Δ' over Ω is a function from Δ to Δ' which preserves standard constants and the type of each constant. The category of signatures over Ω is denoted by $\mathit{HolSign}(\Omega)$. Actually, we have a functor $\mathit{HolSign} : \mathbf{HolTypeSign} \to \mathbf{Cat}$ which lifts each type signature morphism $f : \Omega \to \Omega'$ to a functor $\mathit{HolSign}(f) : \mathit{HolSign}(\Omega) \to \mathit{HolSign}(\Omega')$. A *full signature* $\Omega \triangleright \Delta$ consists of a type signature Ω and a signature Δ over Ω. The category of full signatures is $\sum \Omega : \mathbf{HolTypeSign} . \mathit{HolSign}(\Omega)$ and denoted by $\mathbf{HolFullSign}$.[3]

Given a full type context $\Omega \triangleright \alpha^n : \mathsf{Type}^n$, a *context* over $\Omega \triangleright \alpha^n : \mathsf{Type}^n$ has the form $x^m : \sigma^m$, with distinct variables x_i^m and types σ_i^m over $\Omega \triangleright \alpha^n : \mathsf{Type}^n$. The category of contexts over $\Omega \triangleright \alpha^n : \mathsf{Type}^n$ is denoted by $\mathit{HolCtxt}(\Omega \triangleright \alpha^n : \mathsf{Type}^n)$ and its morphisms, called *context inclusions*, are inclusions between the sets of declared context variables such that the type of each variable is preserved.[4] Actually, we have a functor $\mathit{HolCtxt} : \mathbf{HolFullTypeCtxt} \to \mathbf{Cat}$ lifting full type context morphisms to functors. A *full context* takes the form $\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m$, where Δ is a signature over Ω and where $x^m : \sigma^m$ is a context over $\Omega \triangleright \alpha^n : \mathsf{Type}^n$. The category of *full contexts* is $\mathbf{HolFullCtxt} = \sum \Omega : \mathbf{HolTypeSign} . \mathit{HolSign}(\Omega) \times \sum \omega : \mathbf{HolTypeCtxt} . \mathit{HolCtxt}(\Omega \triangleright \omega)$.

Let $\Omega \triangleright \Delta$ be a full signature. The set $\mathcal{TT}^{Hol}$ of typed terms over $\Omega \triangleright \Delta$ is a subset of expressions of the form $\alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m \triangleright M : \tau$ such that $\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m$ is a full context. It is inductively defined as follows:[5] **(1)** $\alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m \triangleright x_i^m : \sigma_i^m$ is in $\mathcal{TT}^{Hol}$ for each $i \in \{1 \ldots m\}$. **(2)** If $\beta^k : \mathsf{Type}^k \triangleright c : \tau$ is a constant declaration in Δ and $\alpha^n : \mathsf{Type}^n \triangleright \tau_i^k$ is a type over Ω for each $i \in \{1 \ldots k\}$ then $\alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m \triangleright c_{\tau[\tau^k/\beta^k]} : \tau[\tau^k/\beta^k]$ is in $\mathcal{TT}^{Hol}$. **(3)** If $\alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m, y : \rho \triangleright M : \tau$ is in $\mathcal{TT}^{Hol}$, then $\alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m \triangleright (\lambda\, y : \rho\, .\, M) : \rho \to \tau$ is in $\mathcal{TT}^{Hol}$. **(4)** If $\alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m \triangleright M : \rho \to \tau$ and $\alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m \triangleright N : \rho$ are in $\mathcal{TT}^{Hol}$, then $\alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m \triangleright (M\ N) : \tau$ is in $\mathcal{TT}^{Hol}$. If $\alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m \triangleright M : \tau$ is a typed term over $\Omega \triangleright \Delta$ we say that M is a *term* over $\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m$ and τ is its *type*. We will make use of the property of the HOL type system that M has a unique type. The set of *terms* over a full context $\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m$ is denoted by $\mathit{HolTerm}(\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m)$. Each full context morphism is lifted to a function between sets of terms. This defines a functor

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[3] For a functor $F : \mathbf{I} \to \mathbf{Cat}$ the category ΣF, also written $\sum x : \mathbf{I} . F(x)$, is defined by the general Grothendiek construction which generalizes the set-theoretic sum.

[4] More powerful morphisms that allow for renaming of context variables are possible if the notion of term is generalized so that *accidental hiding* of context variables by λ-abstractions is avoided. A possible framework to formalize this is provided by CINNI, the Calculus of Indexed Names and Named Indices [10].

[5] To avoid any technicalities connected with the treatment of bound variables we specialize the notion of terms in HOL by enforcing the convention that all bound and free variables are distinct.

HolTerm : **HolFullCtxt** $\to$ **Set**. On terms we assume the usual notion of α-conversion $\equiv_\alpha$, but we do *not* identify terms that are α-convertible.

A *formula* over a full signature $\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m$ is a term over $\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m$ of type o. A *sentence* over $\Omega \triangleright \Delta$ has the form $\alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m \triangleright A^k \Vdash A$ with formulae A_i^k and A over $\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m$. The set of sentences over $\Omega \triangleright \Delta$ is denoted by $\mathit{HolSen}(\Omega \triangleright \Delta)$. Each signature morphism can be lifted to a function between sets of sentences. So we have a functor HolSen : **HolFullSign** $\to$ **Set**.

3.2 Deduction and Entailment

Given a full signature $\Sigma = (\Omega \triangleright \Delta)$ the *deductive system of HOL* over Σ is the binary relation $(\vdash_1^{Hol}) \subseteq \mathcal{P}_{\mathrm{fin}}(\mathit{HolSen}(\Sigma)) \times \mathit{HolSen}(\Sigma)$ specified by the *inference rules of HOL*, which are the following:

$$\frac{}{\alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m \triangleright A \Vdash A} \qquad \text{(ASSUME)}$$

$$\frac{\alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m \triangleright G \Vdash A \quad \alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m \triangleright H \Vdash A \Rightarrow B}{\alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m \triangleright H, G \Vdash B} \qquad \text{(MP)}$$

$$\frac{\alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m \triangleright H \Vdash B}{\alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m \triangleright G \Vdash A \Rightarrow B} \qquad \text{(DISCH)}$$

where G is obtained from H by removing all formulae α-equivalent to A.

$$\frac{}{\alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m \triangleright \Vdash M =^\rho M} \qquad \text{(REFL)}$$

$$\frac{}{\alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m \triangleright \Vdash ((\lambda\, y : \rho\, .\, M)\, N) =^\tau M[N/y]} \qquad \text{(BETA-CONV)}$$

$$\frac{\alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m \triangleright H_i^k \Vdash M_i^k =^{\rho_i^k} N_i^k \text{ for all } i \in \{1 \ldots k\} \quad \alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m \triangleright G \Vdash A[M^k/z^k]}{\alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m \triangleright H^k, G \Vdash A[N^k/z^k]} \qquad \text{(SUBST)}$$

$$\frac{\alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m, y : \rho \triangleright H \Vdash M =^\tau N}{\alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m \triangleright H \Vdash (\lambda\, y : \rho\, .\, M) =^{\rho \to \tau} (\lambda\, y : \rho\, .\, N)} \qquad \text{(ABS)}$$

where x is not free in H.

$$\frac{\alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m \triangleright H \Vdash A}{\beta^k : \mathsf{Type}^k \triangleright x^m : \sigma^m[\tau^n/\alpha^n] \triangleright H \Vdash A[\tau^n/\alpha^n]} \qquad \text{(INST-TYPE)}$$

where $\beta^k : \mathsf{Type}^k \triangleright \tau_i^n$ are types and α_i^n does not occur in H for $i \in \{1 \ldots n\}$.

For better readability we have written $M =^\rho N$ instead of $(((=_{\rho \to \rho \to \mathsf{o}})\, M)\, N)$.

As in the HOL system the assumptions on the lefthand side of $\Vdash$ are lists instead of sets, which is justified by the fact that HOL inference rules do not depend on their order. Another noteworthy point is that we have not identified α-equivalent terms. α-conversion is, however, used implicitly in the HOL implementation of the rules MP and SUBST, which makes their implementation slighly more liberal than the rules given above. To compensate for this, we assume in addition to the rules above a rule for renaming of bound variables, thereby avoiding a commitment to the technicalities of the HOL implementation.

Given a full signature Σ, the deductive system of HOL defines the *entailment relation* $(\vdash^{Hol}_{\Sigma}) \subseteq \mathcal{P}_{\mathrm{fin}}(\mathit{HolSen}(\Sigma)) \times \mathit{HolSen}(\Sigma)$ of HOL as follows: $\Gamma \vdash^{Hol}_{\Sigma} \phi$ iff ϕ can be derived from Γ in the deductive system of HOL. Derivability of a sentence ϕ from a set of sentences Γ in the deductive system of HOL is defined in the conventional way employing sequential proofs. According to the terminology of [8] (**HolFullSign**, $\mathit{HolSen}, \vdash^{Hol}$) constitutes an entailment system. We call it *entailment system of HOL*.

3.3 Theories

An *(axiomatic) HOL theory* (Σ, Γ) consists of a full signature Σ together with a set Γ of axioms over Σ. A full signature morphism $H : \Sigma \to \Sigma'$ is said to be a *theory morphism* $H : (\Sigma, \Gamma) \to (\Sigma', \Gamma')$ iff $\Gamma' \vdash^{Hol}_{\Sigma'} \mathit{HolSen}(H)(\phi)$ for all $\phi \in \Gamma$. This gives a category of theories that will be denoted by **HolTh**.

All mathematical developments in HOL take place in *standard theories* extending the *logical theory* `bool`. Therefore, for the remainder of this article we define `bool` as `o`, and we use `bool` to emphasize that we are working with classical extensional logic. Indeed, it is the logical theory `bool` which makes HOL a logic of this kind. The logical theory $\texttt{bool} = (\Sigma, \Gamma)$ with $\Sigma = (\Omega, \Delta)$ has in addition to the standard (type) constants the following structure:

Σ contains the constants $\texttt{T} : \texttt{bool}$, $\texttt{F} : \texttt{bool}$, $\neg : \texttt{bool} \to \texttt{bool}$, $\wedge : \texttt{bool} \to \texttt{bool} \to \texttt{bool}$, $\vee : \texttt{bool} \to \texttt{bool} \to \texttt{bool}$, $\forall : (\alpha \to \texttt{bool}) \to \texttt{bool}$, $\exists : (\alpha \to \texttt{bool}) \to \texttt{bool}$ and Γ contains suitable definitional axioms to equip these constants with their standard (classical) meaning. Indeed all the constants above are defined in terms of (higher-order) equality $=$, implication $\Rightarrow$ and Hilbert's ϵ-operator, which is another constant in Σ declared by $\epsilon : (\alpha \to \texttt{bool}) \to \alpha$. Moreover, there are some non-definitional axioms in Γ, namely

$\Vdash\ \forall x : \texttt{bool} \,.\, (x =^{\texttt{bool}} \texttt{T}) \vee (x =^{\texttt{bool}} \texttt{F})$ (BOOL-CASES-AX)

$\Vdash\ \forall x : \texttt{bool} \,.\, \forall y : \texttt{bool} \,.\, (x \Rightarrow y) \Rightarrow (y \Rightarrow x) \Rightarrow (x =^{\texttt{bool}} y)$ (IMP-ANTISYM-AX)

$f : \alpha \to \beta \ \Vdash\ (\lambda\, x : \alpha \,.\, (f\, x)) =^{\beta} f$ (ETA-AX)

$\Vdash\ \forall P : \alpha \to \texttt{bool} \,.\, \forall x : \texttt{bool} \,.\, (P\, x) \Rightarrow (P\, (\epsilon\, P))$ (SELECT-AX)

with the usual abbreviation $\forall x : \sigma \,.\, M$ for $\forall(\lambda\, x : \sigma \,.\, M)$.

Finally, Ω contains an atomic type constant `ind`, and there is a last axiom INFINITY-AX stating that `ind` has an infinite number of elements.

4 Nuprl

Nuprl type theory [1] is a variant of Martin-Löf's 1982 polymorphic, extensional type theory (the version contained in [9] with extensional equality). Although Nuprl has very advanced features (e.g. subset types, subtyping, quotient types, recursive types, intersection types, partial functions, and direct computation) which make these type theories rather different, semantically Nuprl can be viewed as an extension of Martin-Löf's type theory, in the sense that it has a richer variety of types and more flexible rules which give rise to a richer collection of well-typed terms.

Even though the advanced features of Nuprl provide an important motivation for the HOL-Nuprl connection, the connection itself does not rely on features that go beyond Martin-Löf's type theory as presented in [9]. Indeed, in the following we present the part of classical Nuprl that is sufficent to establish the HOL-Nuprl connection. We give a simplified presentation based on [1], but, as for HOL, we have used a categorical formulation in order to achive a uniform and systematic description of the translation.

4.1 Syntactic Categories

We assume countably infinite disjoint sets of *constants* and *variables* with the same meta-variable conventions as for HOL.

A *Nuprl signature* Σ is a set of constants. We assume that Σ contains a number of *standard constants* including $\mathbb{U}_i$ for $i \geq 1$, `app`, λ, `pi` and further constants that we do not explicitly mention here. We denote the category of signatures and bijective functions preserving standard constants by **NuprlSign**.

Given a signature Σ we inductively define the set of *terms* $\mathcal{T}^{Nuprl}$ over Σ as follows: **(1)** $x^m \triangleright x_i^m$ is in $\mathcal{T}^{Nuprl}$ for each $i \in \{1 \ldots m\}$. **(2)** If c is a constant in Δ and $x^m \triangleright M_i^k$ is in $\mathcal{T}^{Nuprl}$ for each $i \in \{1 \ldots k\}$ then $x^m \triangleright c(M^k)$ is in $\mathcal{T}^{Nuprl}$. We write c instead of $c(M^k)$ if $k = 0$. **(3)** If $x^m, y \triangleright M$ is in $\mathcal{T}^{Nuprl}$ then $x^m \triangleright (y.M)$ is in $\mathcal{T}^{Nuprl}$. If $x^m \triangleright M$ is a term over Σ we say that M is a *term* over $\Sigma \triangleright x^m$. Instead of making constants c explicit by writing $c(M^k)$ we often use a more convenient notation such as $(M\ N)$ for $\mathtt{app}(M, N)$, $\lambda\, x\, .\, M$ for $\lambda(x.M)$ and $\Pi\, x : S\, .\, T$ for $\mathtt{pi}(S, (x.T))$.

Given a signature Σ, a *context* over Σ takes the form $x^m : S^m$, where S_{i+1}^m is a term over $\Sigma \triangleright x_1^m, ..., x_i^m$ and where all the variables x_i^m are distinct. A *context inclusion* is an inclusion between the sets of declared context variables such that the type of each variable is preserved. A *full Nuprl context* is of the form $\Sigma \triangleright x^m : S^m$, where $x^m : S^m$ is a context over Σ. The category of contexts over Σ with context inclusions as morphisms is denoted by $NuprlCtxt(\Sigma)$. Furthermore, each signature morphism can be lifted to a morphism between contexts, so that we obtain a functor $NuprlCtxt : \mathbf{NuprlSign} \to \mathbf{Cat}$. The category of full contexts is $\mathbf{NuprlFullCtxt} = \sum\ \Sigma : \mathbf{NuprlSign}\ .\ NuprlCtxt(\Sigma)$.

A *term* over a full context $\Sigma \triangleright x^m : S^m$ is precisely a term over $\Sigma \triangleright x^m$. The set of terms over $\Sigma \triangleright x^m : S^m$ is denoted by $NuprlTerm(\Sigma \triangleright x^m : S^m)$. Each full context morphism is lifted to a function between sets of terms. This defines a functor $NuprlTerm : \mathbf{NuprlFullCtxt} \to \mathbf{Set}$.

Given a signature Σ we also define *sentences* over Σ, which have the form $x^m : S^m \Vdash T$, with a context $x^m : S^m$ over Σ and a term T over $\Sigma \triangleright x^m : S^m$. The set of sentences over Σ is denoted by $NuprlSen(\Sigma)$. For each fixed signature Σ this is a category if we take context inclusions lifted to sentences as morphisms. Moreover, signature morphisms can be lifted to functors between such categories, so that we have a functor $NuprlSen : \mathbf{NuprlSign} \to \mathbf{Cat}$.

It is worthwhile mentioning that this notion of sentence is a proper specialization of the judgements admitted in [1], which take the form $x^m : S^m \Vdash T\,[\mathrm{ext}\,P]$, the pragmatic intention being that the extraction term P is usually hidden from the user, but it can be extracted from a completed proof. Assuming that T is intended as a proposition, we are usually not interested in the extraction term P. Therefore we will only use *abstract judgements* of the form $x^m : S^m \Vdash T$. We define such an abstract judgement to be derivable iff $x^m : S^m \Vdash T\,[\mathrm{ext}\,P]$ is derivable for some P. To make explicit an element P that inhabits a type T, we use equality types of the form $M = N \in T$, so that we can derive $x^m : S^m \Vdash T\,[\mathrm{ext}\,P]$ iff $x^m : S^m \Vdash P \in T$ is derivable, where we assume that $P \in T$ is identified with $P = P \in T$.

4.2 Derivability and Entailment

We define an *entailment relation* $(\vdash^{Nuprl}_{\Sigma}) \subseteq \mathcal{P}_{\mathrm{fin}}(\mathit{NuprlSen}(\Sigma)) \times \mathit{NuprlSen}(\Sigma)$ where $\Gamma \vdash^{Nuprl}_{\Sigma} \phi$ holds iff the sentence ϕ can be derived using the sentences in Γ as assumptions. Here, we consider derivability in the fragment of classical Nuprl given by the inference rules below, which are either basic-inference rules or trivially derivable. (**NuprlSign**, *NuprlSen*, $\vdash^{Nuprl}$) constitutes an entailment system. We call it the *entailment system of Nuprl*.

$$\frac{H, G \Vdash T : \mathbb{U}_i}{H, x : T, G \Vdash x \in T} \qquad \text{(HYP)}$$

$$\frac{H \Vdash S : \mathbb{U}_i \quad H, G \Vdash T}{H, x : S, G \Vdash T} \text{ if } x \text{ is not declared in } H \text{ or } G \qquad \text{(THIN)}$$

$$\frac{H, G \Vdash N \in S \quad H, x : S, G \Vdash T}{H, G \Vdash T[N/x]} \qquad \text{(CUT)}$$

$$\frac{H, x : S \Vdash T : \mathbb{U}_i \quad H \Vdash T[M/x] \quad H \Vdash M = N \in S}{H \Vdash T[N/x]} \qquad \text{(SUBST)}$$

$$\frac{}{H \Vdash \mathbb{U}_i \in \mathbb{U}_j} \text{ if } i < j \qquad \text{(UNIV-INTRO)}$$

$$\frac{H \Vdash M \in \mathbb{U}_i}{H \Vdash M \in \mathbb{U}_j} \text{ if } i < j \qquad \text{(CUMULATIVITY)}$$

$$\frac{H \Vdash T \in \mathbb{U}_i \quad H \Vdash M \in T \quad H \Vdash N \in T}{H \Vdash (M = N \in T) \in \mathbb{U}_i} \qquad \text{(EQUALITY-FORM)}$$

$$\frac{H \Vdash M = N \in T}{H \Vdash N = M \in T} \qquad \text{(EQUALITY-SYM)}$$

$$\frac{H \Vdash K = L \in T \quad H \Vdash L = M \in T}{H \Vdash K = M \in T} \qquad \text{(EQUALITY-TRANS)}$$

$$\frac{H \Vdash A \in \mathbb{U}_i \quad H, x : A \Vdash B \in \mathbb{U}_i}{H \Vdash (x : A \to B) \in \mathbb{U}_i} \qquad \text{(FUN-FORM)}$$

$$\frac{H \Vdash S \in \mathbb{U}_i \quad H, x : S \Vdash M \in T}{H \Vdash (\lambda\, x\, .\, M) \in (x : S \to T)} \qquad \text{(FUN-INTRO)}$$

$$\frac{H \Vdash N \in S \quad H \Vdash M \in (x : S \to T)}{H \Vdash (M\ N) \in T[N/x]} \qquad \text{(FUN-ELIM)}$$

$$\frac{H \Vdash S \in \mathbb{U}_i \quad H, x : S \Vdash M = N \in T}{H \Vdash (\lambda\, x\, .\, M) = (\lambda\, x\, .\, N) \in (x : S \to T)} \qquad \text{(FUN-XI)}$$

$$\frac{H \Vdash N \in S \quad H, x : S \Vdash M \in T}{H \Vdash ((\lambda\, x\, .\, M)\ N) = M[N/x] \in B[N/x]} \qquad \text{(FUN-BETA)}$$

$$\frac{H \Vdash f : (x : A \to B) \quad H \Vdash g : (x : A \to B) \quad H, x : A \Vdash (f\ x) = (g\ x) \in B}{H \Vdash f = g \in (x : A \to B)} \qquad \text{(FUN-EXT)}$$

In addition to these rules we assume a rule for renaming of bound variables, and we make use of standard rules for the *empty type* `Void`, the *singleton type* `Unit` with an element $\bullet$, the *boolean type* $\mathbb{B}$ with elements `tt` and `ff` and elimination operator `if` M `then` N `else` N', the *type of natural numbers* $\mathbb{N}$ with standard operations, the *dependent product types* $x : S \times T$ with a pair constructor $<M, N>$ and elimination operators `1of`(M), `2of`(M), the *disjoint union types* $S + T$ with left and right injections `inl`(M), `inr`(M) and elimination operator `decide`$(M, (x.N), (x.N'))$, and *subset types* $\{x : T \mid A\}$. Finally, we assume a standard constant `inhabited` with the classical axiom scheme given later.

The propositions-as-types interpretation is made explicit using the following logical abbreviations: $\mathbb{P}_i = \mathbb{U}_i$, `False` = `Void`, `True` = `Unit`, $A \wedge B = A \times B$, $A \vee B = A + B$, $\forall x : A\,.\,B = x : A \to B$, and $\exists x : A\,.\,B = x : A \times B$. If x is not free in B we write $A \Rightarrow B$ and $A \to B$ instead of $\forall x : A\,.\,B$ and $x : A \to B$, respectively. Furthermore, $\neg A$ abbreviates $A \Rightarrow$ `False`.

4.3 Classical Extension

The translation described in the next section makes use of Nuprl's operator $\uparrow b =$ `if` b `then True else False` which converts an element of $\mathbb{B}$ into a (propositional) type. For the translation of HOL's equality we wish to define a boolean polymorphic equality using Nuprl's propositional equality, but so far we do not have any means for converting a proposition into a boolean, which amounts to deciding whether a propositional type is inhabited. So we add a standard constant `inhabited` and we assume axioms $\Vdash$ `inhabited` $\in x : \mathbb{U}_i \to x + (x \to$ `Void`$)$ stating that `inhabited`(T) decides if its argument, a type T in $\mathbb{U}_i$, is inhabited, and that it returns an element of T if this is the case. Using the new axioms we can easily define an operator $\downarrow_b P =$ `decide`$($`inhabited`$(P), (x.$`tt`$), (x.$`ff`$))$ that casts a propositional type into a boolean by deciding the proposition.

4.4 Theories

An *(axiomatic) Nuprl theory* (Σ, Γ) consists of a signature Σ together with a set Γ of axioms over Σ. The set of all such Γ over a signature Σ is denoted by $NuprlAxSet(\Sigma)$. A signature morphism $H : \Sigma \to \Sigma'$ is said to be a *theory morphism* $H : (\Sigma, \Gamma) \to (\Sigma', \Gamma')$ iff $\Gamma' \vdash^{Nuprl}_{\Sigma'} NuprlSen(H)(\phi)$ for all $\phi \in \Gamma$. This gives a category of theories that will be denoted by **NuprlTh**.

A theory morphism $H : (\Sigma, \Gamma) \to (\Sigma', \Gamma')$ is called *axiom-preserving* iff *NuprlSen* $(H)(\Gamma) \subseteq \Gamma'$. The subcategory of **NuprlTh** with the objects of **NuprlTh** but morphisms restricted to axiom-preserving morphisms is denoted by $\mathbf{NuprlTh_0}$. This will be the target category of the translation between HOL and Nuprl theories. **NuprlTh**, on the other hand, is used for the second stage of the HOL-Nuprl connection, namely, the interpretation of Nuprl theories.

5 Theory Translation

In accordance with the general logics terminology introduced in [8], which is used as a guideline for the following definitions, the translation from HOL theories to Nuprl theories will be specified by a functor $\Phi : \mathbf{HolFullSign} \to \mathbf{NuprlTh_0}$ which translates HOL signatures into Nuprl theories and a natural transformation $\alpha : \mathit{HolSen} \to \mathit{NuprlSen} \circ \Phi$ which translates HOL sentences into Nuprl sentences.

In order to focus on the essence of the translation we will not deal with the technicalities of translating names and avoiding name clashes. Instead, we assume that all HOL (type) constants and (type) variables are non-standard constants and variables, respectively, in Nuprl. We furthermore assume that there is a countably infinite set of reserved variables in Nuprl that are not used to represent HOL (type) variables.

We define the functor $\Phi : \mathbf{HolFullSign} \to \mathbf{NuprlTh_0}$ by $\Phi(\Sigma) = (\Phi(\Sigma), \Psi(\Sigma))$ with a functor $\Phi : \mathbf{HolFullSign} \to \mathbf{NuprlSign}$ and a (dependent) function $\Psi : \prod\ \Sigma : \mathbf{HolFullSign}\ .\ \mathit{NuprlAxSet}(\Phi(\Sigma))$ (to be defined below).

Applying Φ to a full HOL signature $\Sigma = (\Omega \triangleright \Delta)$ yields a Nuprl theory (Σ', Γ') consisting of the signature part Σ' and the axiom part Γ'.

Here, and in the following, we do not define the behaviour of functors on morphisms explicitly. Actually, there is only one natural and straightforward way to extend our definitions to morphisms.

We deal with the two levels Ω and Δ separately. First, we deal with the type signature Ω, which contains type constants to be translated into a Nuprl signature part $\Phi(\Omega)$ and an axiom part $\Psi(\Omega)$, where $\Phi : \mathbf{HolTypeSign} \to \mathbf{NuprlSign}$ is a functor defined by $\Phi(\{\nu : \mathsf{Type}^n \to \mathsf{Type}\}) = \{\nu\}$, $\Phi(\emptyset) = \emptyset$, $\Phi(\Omega_1 \cup \Omega_2) = \Phi(\Omega_1) \cup \Phi(\Omega_2)$ and $\Psi : \prod\ \Omega : \mathbf{HolTypeSign}\ .\ \mathit{NuprlAxSet}(\Phi(\Omega))$ is a (dependent) function defined by $\Psi(\{\nu : \mathsf{Type}^n \to \mathsf{Type}\}) = \{\alpha^n : \mathsf{S} \Vdash \nu(\alpha^n) \in \mathsf{S}\}$, $\Psi(\emptyset) = \emptyset$, $\Psi(\Omega_1 \cup \Omega_2) = \Psi(\Omega_1) \cup \Psi(\Omega_2)$.

Since semantically all HOL types are assumed to be nonempty, we have employed $x : \mathsf{S}$ as a suggestive notation for $x : \mathbb{U}_1,\ \mathit{inh}_x : \mathtt{Inh}(x)$ where inh_x is a reserved variable that is associated to x only and $\mathtt{Inh}(x)$ is defined as $\exists z : x\ .\ \mathtt{True}$. Likewise, $x \in \mathsf{S}$ is a suggestive notation for $x \in \mathbb{U}_1 \wedge \mathtt{Inh}(x)$.

We also we need to translate HOL type contexts $\alpha^n : \mathsf{Type}^n$ into Nuprl contexts giving a functor $\Phi : \mathbf{HolTypeCtxt} \to \mathbf{NuprlCtxt}$ with $\Phi(\alpha^n : \mathsf{Type}^n) = \alpha^n : \mathsf{S}^n$. This functor can be lifted to a functor $\Phi : \mathbf{HolFullTypeCtxt} \to \mathbf{NuprlFullCtxt}$ with $\Phi(\Omega \triangleright \alpha^n : \mathsf{Type}^n) = \Phi(\Omega) \triangleright \Phi(\alpha^n : \mathsf{Type}^n)$.

To specify the translation Ψ of the signature Δ over Ω which contains constant declarations over Ω we need the following natural transformation $\alpha : \mathit{HolType} \to \mathit{NuprlTerm} \circ \Phi$ translating HOL types into Nuprl types, which are terms in Nuprl: **(1)** $\alpha(\Omega \triangleright \alpha^n : \mathsf{Type}^n)(\alpha_i^n) = \alpha_i^n$, **(2)** $\alpha(\Omega \triangleright \alpha^n : \mathsf{Type}^n)(\circ) = \mathbb{B}$, **(3)** $\alpha(\Omega \triangleright \alpha^n : \mathsf{Type}^n)(\sigma \to \tau) = \alpha(\Omega \triangleright \alpha^n : \mathsf{Type}^n)(\sigma) \to \alpha(\Omega \triangleright \alpha^n : \mathsf{Type}^n)(\tau)$, **(4)** $\alpha(\Omega \triangleright \alpha^n : \mathsf{Type}^n)(\nu(\sigma_1, \ldots, \sigma_n)) = \nu(\alpha(\Omega \triangleright \alpha^n : \mathsf{Type}^n)(\sigma_1), \ldots, \alpha(\Omega \triangleright \alpha^n : \mathsf{Type}^n)(\sigma_n))$.

As a next step we give the translation of constants Δ over a type signature Ω. Again, we have a signature part $\alpha(\Omega)(\Delta)$ and an axiom part $\beta(\Omega)(\Delta)$, where $\alpha : \mathit{HolSign} \to \mathbf{NuprlSign}$ is a natural transformation and β is a type-signature-indexed family of (dependent) functions $\beta(\Omega) : \prod\ \Delta : \mathit{HolSign}(\Omega)\ .\ \mathit{NuprlAxSet}(\alpha(\Omega)(\Delta))$: **(1)** $\alpha(\Omega)(\{\alpha^n : \mathsf{Type}^n \triangleright c : \sigma\}) = \{c\}$, **(2)** $\alpha(\Omega)(\emptyset) = \emptyset$, **(3)** $\alpha(\Omega)(\Delta_1 \cup \Delta_2) = \alpha(\Omega)(\Delta_1) \cup \alpha(\Omega)(\Delta_2)$, **(4)** $\beta(\Omega)(\{\alpha^n : \mathsf{Type}^n \triangleright c : \sigma\}) = \{\alpha^n : \mathsf{S}^n \Vdash c(\alpha^n) \in \alpha(\Omega \triangleright \alpha^n : \mathsf{Type}^n)(\sigma)\}$, **(5)** $\beta(\Omega)(\emptyset) = \emptyset$, **(6)** $\beta(\Omega)(\Delta_1 \cup \Delta_2) = \beta(\Omega)(\Delta_1) \cup \beta(\Omega)(\Delta_2)$.

The previous definitions are lifted to full signatures as Φ : **HolFullSign** $\to$ **NuprlSign** and $\Psi : \prod\ \Sigma : \mathbf{HolFullSign}\ .\ \mathit{NuprlAxSet}(\Phi(\Sigma))$ with $\Phi(\Omega \triangleright \Delta) = \Phi(\Omega) \cup \alpha(\Omega)(\Delta)$ and $\Psi(\Omega \triangleright \Delta) = \Psi(\Omega) \cup \beta(\Omega)(\Delta)$, respectively.

Next, we define the translation of contexts $x^m : \sigma^m$, which depends on type contexts and gives therefore rise to a natural transformation $\alpha : \mathit{HolCtxt} \to \mathit{NuprlCtxt} \circ \Phi$ with $\alpha(\Omega \triangleright \alpha^n : \mathsf{Type}^n)(x^m : \sigma^m) = x^m : \alpha(\Omega \triangleright \alpha^n : \mathsf{Type}^n)(\sigma^m)$.

We now lift the natural transformation α to a functor Φ : **HolFullCtxt** $\to$ **NuprlFullCtxt** between full contexts defined by $\Phi(\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m) = \Phi(\Omega \triangleright \Delta) \triangleright \Phi(\alpha^n : \mathsf{Type}^n), \alpha(\Omega \triangleright \alpha^n : \mathsf{Type}^n)(x^m : \sigma^m)$.

Our goal is the translation of HOL sentences into Nuprl sentences. Since HOL sentences are constructed from terms we define their translation $\alpha : \mathit{HolTerm} \to \mathit{NuprlTerm} \circ \Phi$ first: **(1)** $\alpha(\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m)(x_i^m) = x_i^m$, **(2)** $\alpha(\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m)(=_{\tau \to \tau \to \mathsf{o}}) = (=_{\mathsf{b}}^{\alpha(\Omega)(\tau)})$, **(3)** $\alpha(\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m)(\Rightarrow_{\mathsf{o} \to \mathsf{o} \to \mathsf{o}}) = (\Rightarrow_{\mathsf{b}})$, **(4)** $\alpha(\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m)(c_{\tau[\tau^k/\beta^k]}) = c(\alpha(\Omega \triangleright \alpha^n : \mathsf{Type}^n)(\tau_1^k), \ldots, \alpha(\Omega \triangleright \alpha^n : \mathsf{Type}^n)(\tau_k^k))$, where $(\beta^k : \mathsf{Type}^k \triangleright c : \tau) \in \Delta$ and $\alpha^n : \mathsf{Type}^n \triangleright \tau_i^k$ for $i \in \{1...k\}$ are types over Ω, **(5)** $\alpha(\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m)(M\ N)) = (\alpha(\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m)(M)\ \alpha(\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m)(N))$, **(6)** $\alpha(\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m)(\lambda\ y : \rho\ .\ M) = (\lambda\ y\ .\ \alpha(\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m, y : \rho)(M))$ where we employed the abbreviations $\Rightarrow_{\mathsf{b}}$ for $\lambda\, x, y\ .\ \mathtt{if}\ x\ \mathtt{then}\ y\ \mathtt{else}\ \mathtt{tt}$, and $=_{\mathsf{b}}^T$ for $\lambda\, x, y\ .\ \downarrow_b(x = y \in T)$.

Now the natural transformation $\alpha : \mathit{HolSen} \to \mathit{NuprlSen} \circ \Phi$ of HOL sentences is given by $\alpha(\Omega \triangleright \Delta)(\alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m \triangleright H \Vdash A) = \Phi(\alpha^n : \mathsf{Type}^n), \alpha(\Omega \triangleright \alpha^n : \mathsf{Type}^n)(x^m : \sigma^m) \triangleright \alpha(\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m)(H) \Vdash \uparrow \alpha(\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m)(A)$ with **(1)** $\alpha(\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m)([]) = []$ **(2)** $\alpha(\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m)([A_1, \ldots, A_n]) = [\ h_1 : \uparrow \alpha(\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m)(A_1), \ldots, h_n :$

$\uparrow\alpha(\Omega \triangleright \Delta \triangleright \alpha^n : \mathsf{Type}^n \triangleright x^m : \sigma^m)(A_n)\ ]$. Here $h_1, \ldots, h_n$ are reserved variables of Nuprl distinct from HOL variables and not used for any other purpose.

6 Correctness of the Translation

In the following we adapt some terminology from the theory of general logics [8]. To justify the translation we introduce an appropriate notion of map between entailment systems. For sake of brevity, we will immediately instantiate the general notions to the specific categories and functors we are interested in.

First we observe that the functor Φ : **HolFullSign** $\rightarrow$ **NuprlTh$_0$** can be easily extended to a functor Φ : **HolTh$_0$** $\rightarrow$ **NuprlTh$_0$**, called the *α-extension to theories*, by mapping an HOL-theory (Σ, Γ) to the Nuprl theory (Σ', Γ') with $\Sigma' = \Phi(\Sigma)$ and $\Gamma' = \Psi(\Sigma) \cup \alpha_\Sigma(\Gamma)$.

We also extend the functors *HolSen* : **HolFullSign** $\rightarrow$ **Cat** and *NuprlSen* : **NuprlSign** $\rightarrow$ **Cat** to functors *HolSen* : **HolTh$_0$** $\rightarrow$ **Cat** and *NuprlSen* : **NuprlTh$_0$** $\rightarrow$ **Cat** by just ignoring the axioms in the theories.

Correspondingly, we extend the natural transformation α : *HolSen* $\rightarrow$ *NuprlSen*$\circ$ Φ with *HolSen* : **HolFullSign** $\rightarrow$ **Cat** and *NuprlSen* : **NuprlSign** $\rightarrow$ **Cat** to a natural transformation between *HolSen* : **HolTh$_0$** $\rightarrow$ **Cat** and *NuprlSen* $\circ \Phi$: **HolTh$_0$** $\rightarrow$ **Cat** by setting $\alpha_{(\Sigma,\Gamma)} = \alpha_\Sigma$.

Now Φ : **HolTh$_0$** $\rightarrow$ **NuprlTh$_0$** is an *α-simple* functor [8], i.e., it is the α-extension to theories of a functor from signatures to theories. However, since the axiomatic part in the target of the translation is essential, Φ is *not* α-plain [8], i.e., it is *not* the α-extension of a functor that factors through a functor F : **HolFullSign** $\rightarrow$ **NuprlSign** mapping a signature to a theory with an empty set of axioms.

Correctness Theorem

(Φ, α) : (**HolFullSign**, *HolSen*, $\vdash^{Hol}$) $\rightarrow$ (**NuprlSign**, *NuprlSen*, $\vdash^{Nuprl}$) is a *map between entailment systems*, i.e., the following properties are satisfied:

1. α : *HolSen* $\rightarrow$ *NuprlSen* $\circ \Phi$ is a natural transformation,
2. Φ : **HolTh$_0$** $\rightarrow$ **NuprlTh$_0$** is α-sensible (see below),
3. $\Gamma \vdash^{Hol}_{\Sigma} \phi \Rightarrow \Psi(\Sigma) \cup \alpha_\Sigma(\Gamma) \vdash^{Nuprl}_{\Phi(\Sigma)} \alpha_\Sigma(\phi)$ for each full HOL signature Σ. As before $\Psi(\Sigma)$ denotes the axiom part of $\Phi(\Sigma)$.

Proof Sketch

Concerning (1) we have not made explicit the translation of morphisms. It should, however, be clear that there is only one natural choice and α is a natural transformation, since it is compatible with all renaming morphisms.

In condition (2) Φ : **HolTh$_0$** $\rightarrow$ **NuprlTh$_0$** is called α-sensible, iff: (a) there is a functor $\Phi^\diamond$: **HolFullSign** $\rightarrow$ **NuprlSign** such that *NuprlSign* $\circ \Phi = \Phi^\diamond \circ$ *HolFullSign*, where *HolFullSign* : **HolTh$_0$** $\rightarrow$ **HolFullSign** and *NuprlSign* : **NuprlTh$_0$** $\rightarrow$ **HolFullSign** are the obvious forgetful functors, and (b) $\Gamma'^\bullet =$

$(\emptyset' \cup \alpha_\Sigma(\Gamma))^\bullet$ whenever $\Phi(\Sigma,\emptyset) = (\Sigma',\emptyset')$ and $\Phi(\Sigma,\Gamma) = (\Sigma',\Gamma')$. Here $\Gamma^\bullet = \{\phi : \Gamma \vdash \phi\}$ is the *deductive closure* where $\vdash$ refers to $\vdash^{Hol}$ or $\vdash^{Nuprl}$.

This is clearly satisfied in our case (chosing Φ : **HolFullSign** $\rightarrow$ **NuprlSign** for $\Phi^\diamond$). Actually, the fact that Φ is simple is a stronger property (we have $\Gamma' = \emptyset' \cup \alpha_\Sigma(\Gamma)$) which shows that we do not need the full generality of maps between entailment systems here.

In order to verify (3), namely that the entailment relation is preserved, it is sufficient to verify that the translations of HOL inference rules can be derived in Nuprl assuming the axioms $\Psi(\Sigma)$. To minimize meta-logical reasoning we use of a number of simple theorems that have been verified inside Nuprl:

```
*T assert_bequal_1 ∀T:𝕌. ∀x,y:T.  x = y ⇒ ↑(x =ᵇ y)
*T assert_bequal_2 ∀T:𝕌. ∀x,y:T.  ↑(x =ᵇ y) ⇒ x = y
*T REFL_lemma      ∀T:𝕌. ∀t:T.  ↑(t =ᵇ t)
*T SUBST_lemma     ∀T:𝕌. ∀a,b:T. ∀p:T → 𝕌.  ↑(a =ᵇ b) ⇒ (p a) ⇒ (p b)
*T DISCH_lemma     ∀p,q:𝔹.  (↑p ⇒ ↑q) ⇒ ↑(p ⇒ᵇ q)
*T MP_lemma        ∀t1,t2:𝔹.  ↑(t1 ⇒ᵇ t2) ⇒ ↑t1 ⇒ ↑t2
```

Indeed, using these theorems, most of the translated inference rules have surprisingly short proofs in Nuprl. For details we refer to the extended version of this paper which is available via WWW at `www.csl.sri.com/~stehr`. Even though meta-logical reasoning can be minimized by using the theorems above, the metalogical nature of these proofs is unavoidable. The proof translator to be discussed in Section 8 can be seen as a formalization of their computational content using the metalanguage of the Nuprl system, which is ML.

7 Nuprl Theory Interpretations

Typically, we are not interested in the full generality of using an axiomatic Nuprl theory which has been obtained by translation from an axiomatic HOL theory. Instead, we may prefer to *instantiate constants* with fixed meanings. One reason may be that we want them to be *computationally meaningful*.[6] This (partial) instantiation of constants leads from one axiomatic Nuprl theory to another one, typically with a more restricted class of models. Usually, some or all of the original axioms become theorems in the new theory. Ideally, we obtain a definitional theory, which means that we have constructed an *internal model* of an axiomatic Nuprl theory inside Nuprl itself.

Given Nuprl theories (Σ,Γ) and (Σ',Γ') we say that (Σ,Γ) is *interpreted* in (Σ',Γ') by I iff $I : (\Sigma,\Gamma) \rightarrow (\Sigma',\Gamma')$ is a theory morphism in the category **NuprlTh**. Notice that these morphisms are not necessarily axiom-preserving, since it is typically the point of such an interpretation to get rid of axioms.

Since we have to verify $\Gamma' \vdash^{Nuprl}_{\Sigma'} \mathit{NuprlSen}(I)(\phi)$ for all $\phi \in \Gamma$, the sentences $\mathit{NuprlSen}(I)(\phi)$ are called *proof obligations*. As explained in the beginning, the

[6] For instance, the fact that a new datatype is introduced in HOL by a conservative theory extensions that characterizes the datatype only up to isomorphism gives us some freedom for its interpretation in Nuprl.

activity of setting up a theory morphism and verifying the proof obligations characterizes the second stage of the HOL-Nuprl connection which requires user interaction in general. Subsequently, we interpret the translation of the logical theory of HOL. In fact, we will see that this can be done, thanks to the classical axiom scheme for `inhabited`, in a purely definitional way.

7.1 Interpreting the Logical Theory

The logical HOL theory `bool` is a theory like every other theory, and does not need any particular consideration. It is however important to notice that `bool` is not a pure definitional theory in HOL. Still, it does *not* require any external meta-logical treatment, since all its axioms can be be verified in a rigorous formal way *inside* the classical Nuprl system. We follow [7] where the proof obligations have been verified using the interpretation given next.

Each HOL constant c will be interpreted by a Nuprl constant $[\![c]\!]$. For the logical constants we use assume the following definitional axioms: **(1)** $[\![\neg]\!] = \lambda\ x\ .$ `if` x `then ff else tt`, **(2)** $[\![\wedge]\!] = \lambda\ x, y\ .$ `if` x `then` y `else ff`, **(3)** $[\![\vee]\!] = \lambda\ x, y\ .$ `if` x `then tt else` y, **(4)** $[\![\forall]\!] = \lambda\ T\ .\ \lambda\ P\ .\ \downarrow_b(\forall x : T\ .\ \uparrow P)$, **(5)**, $[\![\exists]\!] = \lambda\ T\ .\ \lambda\ P\ .\ \downarrow_b(\exists x : T\ .\ \uparrow P)$.

As an abbreviation we use $\texttt{arb}(T) = \texttt{decide}(\texttt{inhabited}(T), x.x, x.\bullet)$ which picks an arbitrary element inhabiting a nonempty type T. Hilbert's epsilon operator ($[\![\epsilon]\!]\ T\ P$), where P is a boolean predicate on M, picks an element of the subset of T specified by P if this subset is nonempty, or yields an arbitrary element of T otherwise. This motivates adding a definitional axiom $[\![\epsilon]\!] = \lambda\ T\ .\ \lambda\ P\ .\ \texttt{decide}(\{x : T \mid (P\ x)\}, x.x, x.\texttt{arb}(T))$.

Finally, the HOL datatype `ind` is interpreted by $[\![\texttt{ind}]\!] = \mathbb{N}$, where $\mathbb{N}$ is the Nuprl datatype of natural numbers.

Using the interpretation $[\![\_]\!]$ all the proof obligations, i.e., the translated axioms of the HOL theory `bool` can be discharged and only the definitional Nuprl axioms given above will remain. The fact that HOL types are translated to nonempty Nuprl types is essential to verify the proof obligation corresponding to the declaration of ϵ.

8 Towards a Proof Translator

The translation descibed in this paper is a translation between theories, i.e., we translate a HOL sentence A over Σ into a Nuprl sentence $\alpha(A)$ over $\Phi(\Sigma)$. The proof-theoretic correctness result ensures that if A is provable from Γ in HOL then $\alpha(A)$ is provable from $\Psi(\Sigma) \cup \alpha_\Sigma(\Gamma)$ in Nuprl. The proof of the correctness theorem is carried out by constructing a Nuprl proof of $\alpha(A)$ for each HOL proof of A. This suggests extending of the translation between theories to a translation between proof calculi [8] so that this proof translation is a computable function between the datatypes of proofs. Indeed, we have found that this idea is feasible in practice as demonstrated by the implementation of an HOL-Nuprl proof translator that will be briefly described in this section.

The proof translator works together with HOL90.10 and has been implemented in ML on top of the Nuprl system. This facilitates the translation of HOL proofs into Nuprl proofs considerably, since the powerful collection of Nuprl tactics can be employed. Most of the inference rules of the HOL logic can be handled by a simple combination of Nuprl tactics and the theorems given earlier in a very direct way. Routine proof obligations such as well-typedness conditions are discharged automatically by Nuprl.

In spite of the clear theoretical picture from which we started we had to overcome some difficulties in the implementation that are worth mentioning:

1. The first difficulty is of a technical nature, namely, that the HOL system is designed in such a way that proof objects are not explicitly present. Consequently, it was necessary to equip the system with an an explict notion of proof object that can be extracted from each theorem.
2. Another problem is caused by the somewhat unsatisfactory situation that the HOL system does not only use the proof rules of the HOL logic as presented in the HOL documentation, but it uses in addition a considerable number of derived rules that are not reduced to basic proof rules. We guess that efficiency considerations are the reason for this "optimization", but for us this means that either:
   (a) the derived rules have to be eliminated *inside* HOL by reducing them to basic inference rules, or
   (b) that the derived rules must be given the same status as basic inference rules, which increases the number of rules that have to be treated *outside* HOL by the proof translator.

   In the long term we plan to realize alternative (b), since it turned out that, as in the case of the basic inference rules, most of the derived rules mentioned above can be implemented in Nuprl with little effort. At present, we support nearly all basic proof rules and a number of derived rules.
3. Finally, the proof translator can also deal with another particularity: The HOL implementation of the MP rule also applies to a premise $\neg M$ presupposing that $\neg M$ is equal to $M \Rightarrow \mathtt{F}$. In other words, the definitional axiom of the logical theory is built into the inference rules.

So far the proof translator has been applied to a number of simple HOL theorems of logical nature, namely, to all theorems contained in `taut_thms.sml`. Altough these theorems are quite simple, this is definitely not true for the extracted HOL proofs. The translation of the proof of `DE_MORGAN_THM`, which was automatically generated by HOL, revealed that the proof objects generated by HOL can be surprisingly large and confusing. In this case the HOL proof contained 4733 HOL inference rules. We expect that this problem occurs only if proofs are generated by certain automatic brute-force tactics, and that it can be circumvented by optimizing these tactics in HOL or by proving such theorems using more efficient tactics in Nuprl. Nevertheless these proofs were good test cases for the capabilities of the translator and also for the capabilities of Nuprl to handle large proof objects.

9 Concluding Remarks

We have complemented Howe's semantics-based justification of the HOL-Nuprl connection with a proof-theoretic counterpart. Our result showing that the translation can be seen as a map between the entailment systems of HOL and Nuprl does not only provide a simple proof-theoretic justification for *translating theories*, but it does in addition suggest *proof translation* as an interesting application that has not been considered in the context of the HOL-Nuprl connection so far. To explore the feasibility of this approach we have implemented a proof translator on top of Nuprl. Our experiments with a number of simple theorems indicate that proof translation is practical even though the proof objects can be rather complex. As a consequence we think that scaling up the approach to complex libraries of formal mathematics is a realistic goal, provided that proof objects are not flat proofs but *structured proofs* that reflect the modularity of the development in terms of intermediate lemmas.

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