Techical Appendix to The Production Pipeline and Aggregate Fluctuations

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The equilibrium of the economy detailed in Section 3 of the paper consists of a set of functions defining the behavior of consumption, investment, output, etc., as functions of the model's exogenous forcing variables and capital and inventory stocks. For calculational ease, some additional variables are introduced. These include lagged labor services, $LN_{t+1}^{S,j} = N_t^{S,j}$ for each sector, lagged inventories, $Lv_{t+1}^{j} = v_t^{j}$ for the primary and intermediate sectors, and value added for the intermediate and final industries:

$$\begin{aligned} va_t^M &= \left(k_t^M\right)^{\xi}\left(N_t^{S,M}\right)^{1-\xi}, \\ va_t^F &= \left(k_t^F\right)^{\alpha}\left(N_t^{S,F}\right)^{1-\alpha}. \end{aligned}$$

The model is solved by considering the problem faced by a social planner who chooses a sequence $\{c_t, L_t, N_t^P, N_t^M, N_t^F, w_t, N_t^{S,P}, N_t^{S,M}, N_t^{S,F}, LN_{t+1}^{S,P}, LN_{t+1}^{S,M}, LN_{t+1}^{S,F}, i_t^P, i_t^M, i_t^F, k_{t+1}^P, k_{t+1}^M, k_{t+1}^F,$ $va_t^M, va_t^F, v_{t+1}^P, v_{t+1}^M, Lv_{t+t}^P, Lv_{t+1}^M, \Theta_t^P, \Theta_t^M, \Theta_t^F, \Phi_t^P, \Phi_t^M, \Phi_t^F, \lambda_t^P, \lambda_t^M, \lambda_t^F, \Gamma_t^P, \Gamma_t^M, \Lambda_t^M, \Lambda_t^F, \Xi_t^P, \Xi_t^M\}$ to maximize:

$$\begin{aligned} \pounds \;=\; & E_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ \frac{1}{1-\sigma}\left(c_t\right)^{1-\sigma} L_t^{\chi} \\ & + w_t\left[1 - L_t - N_t^P - N_t^M - N_t^F\right] \\ & + \sum_{j=\{P,M,F\}} \Theta_t^j \left[LN_{t+1}^{S,j} - \vartheta\left(N_t^j / LN_t^{S,j}\right) LN_t^{S,j}\right] \\ & + \sum_{j=\{P,M,F\}} \Phi_t^j \left[LN_{t+1}^{S,j} - N_t^{S,j}\right] \\ & + \sum_{j=\{P,M,F\}} \lambda_t^j \left[\gamma k_{t+1}^j - (1-\delta) k_t^j - \phi\left(i_t^j / k_t^j\right) k_t^j\right] \\ & + p_t \left[\begin{array}{c} a_t^F\left[\left(va_t^F\right)^{-\varepsilon} + \tau\left(v_t^M\right)^{-\varepsilon}\right]^{-\frac{1}{\varepsilon}} - \\ c_t - i_t^F - \left[i_t^M + \left(v_{t+1}^M - Lv_{t+1}^M\right)\right] - \left[i_t^P + \left(v_{t+1}^P - Lv_{t+1}^P\right)\right] \end{array} \right] \end{aligned}$$

$$
\begin{aligned}
&+\Gamma_t^P\left[a_t^P\left(k_t^P\right)^\theta\left(N_t^{S,P}\right)^\eta\overline{T}^{1-\theta-\eta}-v_{t+1}^P\right]\\
&+\Gamma_t^M\left[a_t^M\left[\left(va_t^M\right)^{-\psi}+\tau\left(v_t^P\right)^{-\psi}\right]^{-\frac{1}{\psi}}-v_{t+1}^M\right]\\
&+\Lambda_t^M\left[\left(k_t^M\right)^\xi\left(N_t^{S,M}\right)^{1-\xi}-va_t^M\right]\\
&+\Lambda_t^F\left[\left(k_t^F\right)^\alpha\left(N_t^{S,F}\right)^{1-\alpha}-va_t^F\right]\\
&+\sum_{j=\{P,M\}}\Xi_t^j\left[Lv_{t+1}^j-v_t^j\right]\Bigg\}
\end{aligned}
$$

$w_t, p_t, \Theta_t^P, \Theta_t^M, \Theta_t^F, \Phi_t^P, \Phi_t^M, \Phi_t^F, \Xi_t^P, \Xi_t^M, \lambda_t^P, \lambda_t^M, \lambda_t^F, \Gamma_t^P, \Gamma_t^M, \Lambda_t^M$, and Λ_t^F are Lagrange multipliers, and are interpreted as utility-denominated shadow prices; w_t and p_t are the real wage and price level, respectively; Θ^P, Θ^M, and Θ^F are the marginal products of labor services; Φ^P, Φ^M, and Φ^F are the marginal products of labor; Ξ_t^P and Ξ_t^M are the marginal products of inventories; λ^P, λ^M, and λ^F are the marginal products of capital; Γ_t^P and Γ_t^M are the expected future prices inventories; and Λ_t^M and Λ_t^M are the prices of value added. The first-order conditions for an interior solution are:

$$(c_t):\quad c_t^{-\sigma}L_t^\chi-p_t=0 \tag{A.1}$$

$$(L_t):\quad \tfrac{\chi}{1-\sigma}c_t^{1-\sigma}L_t^{\chi-1}-w_t=0 \tag{A.2}$$

$$\left(N_t^P\right):\quad -w_t-\Theta_t^P\vartheta'\left(\tfrac{N_t^P}{LN_t^{S,P}}\right)=0 \tag{A.3}$$

$$\left(N_t^M\right):\quad -w_t-\Theta_t^M\vartheta'\left(\tfrac{N_t^M}{LN_t^{S,M}}\right)=0 \tag{A.4}$$

$$\left(N_t^F\right):\quad -w_t-\Theta_t^F\vartheta'\left(\tfrac{N_t^F}{LN_t^{S,F}}\right)=0 \tag{A.5}$$

$$(w_t):\quad 1-L_t-N_t^P-N_t^M-N_t^F=0 \tag{A.6}$$

$$\left(N_t^{S,P}\right):\quad -\Phi_t^P+\eta\Gamma_t^Pa_t^P\left(k_t^P\right)^\theta\left(N_t^{S,P}\right)^{\eta-1}\overline{T}^{1-\theta-\eta}=0 \tag{A.7}$$

$$\left(N_t^{S,M}\right):\quad -\Phi_t^M+(1-\xi)\,\Lambda_t^M\left(k_t^M\right)^\xi\left(N_t^{S,M}\right)^{-\xi}=0 \tag{A.8}$$

$$\left(N_t^{S,F}\right):\quad -\Phi_t^F+(1-\alpha)\,\Lambda_t^F\left(k_t^F\right)^\alpha\left(N_t^{S,F}\right)^{-\alpha}=0 \tag{A.9}$$

$$\left(LN_{t+1}^{S,P}\right): \quad \Theta_t^P + \Phi_t^P - \beta E \Theta_{t+1}^P \left[\vartheta\left(N_{t+1}^P / LN_{t+1}^{S,P}\right) - \left(N_{t+1}^P / LN_{t+1}^{S,P}\right)\vartheta'\left(N_{t+1}^P / LN_{t+1}^{S,P}\right)\right] = 0 \tag{A.10}$$

$$\left(LN_{t+1}^{S,M}\right): \quad \Theta_t^M + \Phi_t^M - \beta E \Theta_{t+1}^M \left[\vartheta\left(N_{t+1}^M / LN_{t+1}^{S,M}\right) - \left(N_{t+1}^M / LN_{t+1}^{S,M}\right)\vartheta'\left(N_{t+1}^M / LN_{t+1}^{S,M}\right)\right] = 0 \tag{A.11}$$

$$\left(LN_{t+1}^{S,F}\right): \quad \Theta_t^F + \Phi_t^F - \beta E \Theta_{t+1}^F \left[\vartheta\left(N_{t+1}^F / LN_{t+1}^{S,F}\right) - \left(N_{t+1}^F / LN_{t+1}^{S,F}\right)\vartheta'\left(N_{t+1}^F / LN_{t+1}^{S,F}\right)\right] = 0 \tag{A.12}$$

$$\left(i_t^P\right): \quad -\lambda_t^P \phi'\left(i_t^P / k_t^P\right) - p_t = 0 \tag{A.13}$$

$$\left(i_t^M\right): \quad -\lambda_t^M \phi'\left(i_t^M / k_t^M\right) - p_t = 0 \tag{A.14}$$

$$\left(i_t^F\right): \quad -\lambda_t^F \phi'\left(i_t^F / k_t^F\right) - p_t = 0 \tag{A.15}$$

$$\left(k_{t+1}^P\right): \quad \gamma\lambda_t^P - \beta E \left\{ \begin{array}{c} \lambda_{t+1}^P \left[(1-\delta) + \phi\left(i_{t+1}^P / k_{t+1}^P\right) - \left(i_{t+1}^P / k_{t+1}^P\right)\phi'\left(i_{t+1}^P / k_{t+1}^P\right)\right] - \\ \theta \Gamma_{t+1}^P a_{t+1}^P \left(k_{t+1}^P\right)^{\theta-1} \left(N_{t+1}^{S,P}\right)^{\eta} \overline{T}^{1-\theta-\eta} \end{array} \right\} = 0 \tag{A.16}$$

$$\left(k_{t+1}^M\right): \quad \gamma\lambda_t^M - \beta E \left\{ \begin{array}{c} \lambda_{t+1}^M \left[(1-\delta) + \phi\left(i_{t+1}^M / k_{t+1}^M\right) - \left(i_{t+1}^M / k_{t+1}^M\right)\phi'\left(i_{t+1}^M / k_{t+1}^M\right)\right] - \\ \xi \Lambda_{t+1}^M a_{t+1}^M \left(k_{t+1}^M\right)^{\xi-1} \left(N_{t+1}^{S,M}\right)^{1-\xi} \end{array} \right\} = 0 \tag{A.17}$$

$$\left(k_{t+1}^F\right): \quad \gamma\lambda_t^F - \beta E \left\{ \begin{array}{c} \lambda_{t+1}^F \left[(1-\delta) + \phi\left(i_{t+1}^F / k_{t+1}^F\right) - \left(i_{t+1}^F / k_{t+1}^F\right)\phi'\left(i_{t+1}^F / k_{t+1}^F\right)\right] - \\ \alpha \Lambda_{t+1}^F a_{t+1}^F \left(k_{t+1}^F\right)^{\alpha-1} \left(N_{t+1}^{S,F}\right)^{1-\alpha} \end{array} \right\} = 0 \tag{A.18}$$

$$\left(va_t^M\right): \quad \Gamma_t^M a_t^M \left(va_t^M\right)^{-\psi-1} \left[\left(va_t^M\right)^{-\psi} + \tau\left(v_t^P\right)^{-\psi}\right]^{-\frac{1}{\psi}-1} - \Lambda_t^M = 0 \tag{A.19}$$

$$\left(va_t^F\right): \quad p_t a_t^F \left(va_t^F\right)^{-\varepsilon-1} \left[\left(va_t^F\right)^{-\varepsilon} + \tau\left(v_t^M\right)^{-\varepsilon}\right]^{-\frac{1}{\varepsilon}-1} - \Lambda_t^F = 0 \tag{A.20}$$

$$\left(v_{t+1}^{P}\right): \quad -p_t - \Gamma_t^{P} + \beta E\left\{\tau \Gamma_{t+1}^{M} a_{t+1}^{M}\left(v_{t+1}^{P}\right)^{-\psi-1}\left[\left(va_{t+1}^{M}\right)^{-\psi} + \tau\left(v_{t+1}^{P}\right)^{-\psi}\right]^{-\frac{1}{\psi}-1} - \Xi_{t+1}^{P}\right\} = 0 \tag{A.21}$$

$$\left(v_{t+1}^{M}\right): \quad -p_t - \Gamma_t^{M} + \beta E\left\{\tau p_{t+1} a_{t+1}^{F}\left(v_{t+1}^{M}\right)^{-\varepsilon-1}\left[\left(va_{t+1}^{F}\right)^{-\varepsilon} + \tau\left(v_{t+1}^{M}\right)^{-\varepsilon}\right]^{-\frac{1}{\varepsilon}-1} - \Xi_{t+1}^{M}\right\} = 0 \tag{A.22}$$

$$\left(Lv_{t+1}^{P}\right): \quad p_t + \Xi_t^{P} = 0 \tag{A.23}$$

$$\left(Lv_{t+1}^{M}\right): \quad p_t + \Xi_t^{M} = 0 \tag{A.24}$$

$$\left(\Theta_t^{j}\right): \quad LN_{t+1}^{S,j} - \vartheta\left(N_t^{j}/LN_t^{S,j}\right)LN_t^{S,j} = 0 \tag{A.25}$$

$$\left(\Phi_t^{j}\right): \quad LN_{t+1}^{S,j} - N_t^{S,j} = 0 \tag{A.26}$$

$$(p_t): \quad \begin{gathered} a_t^{F}\left[\left(va_t^{F}\right)^{-\varepsilon} + \tau\left(v_t^{M}\right)^{-\varepsilon}\right]^{-\frac{1}{\varepsilon}} - c_t - i_t^{F} - \\ \left[i_t^{M} + \left(v_{t+1}^{M} - Lv_{t+1}^{M}\right)\right] - \left[i_t^{P} + \left(v_{t+1}^{P} - Lv_{t+1}^{P}\right)\right] = 0 \end{gathered} \tag{A.27}$$

$$\left(\lambda_t^{j}\right): \quad \gamma k_{t+1}^{j} - (1-\delta)k_t^{j} - \phi\left(i_t^{j}/k_t^{j}\right)k_t^{j} = 0 \tag{A.29}$$

$$\left(\Gamma_t^{P}\right): \quad a_t^{P}\left(k_t^{P}\right)^{\theta}\left(N_t^{S,P}\right)^{\eta}\overline{T}^{1-\theta-\eta} - v_{t+1}^{P} = 0 \tag{A.30}$$

$$\left(\Gamma_t^{M}\right): \quad a_t^{M}\left[\left(va_t^{M}\right)^{-\psi} + \tau\left(v_t^{P}\right)^{-\psi}\right]^{-\frac{1}{\psi}} - v_{t+1}^{M} = 0 \tag{A.31}$$

$$\left(\Lambda_t^{M}\right): \quad \left(k_t^{M}\right)^{\xi}\left(N_t^{S,M}\right)^{1-\xi} - va_t^{M} = 0 \tag{A.32}$$

$$\left(\Lambda_t^{F}\right): \quad \left(k_t^{F}\right)^{\alpha}\left(N_t^{S,F}\right)^{1-\alpha} - va_t^{F} = 0 \tag{A.33}$$

$$\left(\Xi_t^{P}\right): \quad Lv_{t+1}^{P} - v_t^{P} = 0 \tag{A.34}$$

$$\left(\Xi_t^{M}\right): \quad Lv_{t+1}^{M} - v_t^{M} = 0 \tag{A.35}$$

This completes the description of the equations describing the model's solution. The model's solution proceeds in two steps. First, the set of efficiency conditions, constraints, and identities describing this economy is linearized by taking a first-order Taylor series approximation around the model's steady state. This yields a system in which the variables are expressed as percentage deviations from steady state. That is, $\widehat{m}_t = \left(m_t - \overline{m}\right)/\overline{m}$, where $\overline{m}$ is the steady state value of m. For small percentage deviations, $\widehat{m}_t \simeq \ln\left(m_t\right) - \ln\left(\overline{m}\right)$, the data generated by the model can be compared to actual logged data. The resulting linear system was then solved using the King and Watson (1995, 1998) algorithms.

References

[1] King, R. and M. Watson (1995). "System Reduction and Solution Algorithms for Singular Linear Difference Systems Under Rational Expectations." manuscript. University of Virginia.

[2] ____________ and ____________ (1998). "The Solution of Singular Linear Difference Systems Under Rational Expectations." *International Economic Review 39*, 1015-1026.